Maths in *The Walking Dead*

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1 Presentation of the problem

The Walking Dead is a TV serial which tells the story of a Zombie Apocalypse. Zombie stories are classic themes in science fiction, often consistent with infectious disease biology. In the episodes of the serial *The Walking Dead*, or films such as *World War Z* and *Resident Evil*, the infection is caused by virus. In the novel *Deck Z* contagion is caused by bacteria, and prions are the cause in the film *Zombieland*.

Zombism is a deadly disease that kills every infected human and turns the host into a deadly vector of the disease. Our challenge is

- to develop a model of the interaction between humans and zombies;
- to study the model using computer simulations to identify the values of parameters, consistently with the observations based on the TV series *The Walking Dead*;
- to use the model to make predictions about survival of humanity and know how many people will survive to a zombie attack.

The problem Let us assume that an outbreak of zombie infection hits the human population of a nation.

Every day a human will be in one of the following three states:

- the state of susceptible or uninfected humans;

- the infected state of zombies;
- the removed state of a human died for natural causes or a zombie killed by an uninfected human.

Two transitions are possible: a human can become infected by a zombie, and a zombie can be destroyed by a human. Two parameters govern the two transitions:

- a parameter beta gives the rate at which a zombie will infect humans
- another parameter kappa gives the rate that a human kills the zombies.

Could you predict the future of humanity? Under which conditions will humans survive? Is it possible that both humans and zombies survive? Is there any situation in which both humans and zombies are exterminated? May zombies remain constant in time?

2 Introduction to the conjectures and to the obtained results

We decided to study the problem using *discrete mathematics* only. Though, we managed to obtain the analytical expressions which describe the number of humans and of zombies in time, using it to find under which conditions zombie win over humans (or *vice versa*). We also found some particular cases of equilibrium.

3 The *HZR* model

Let us suppose that the possible *states* for each individual of the population are three (Figure 1):

- **human** (H): healthy individual (a common human), who can be infected by a zombie, thus becoming a zombie himself;
- **zombie** (Z): infected individual, who can spread the disease among humans, but might be killed by them;
- **removed** (R): individual who may neither infect nor be infected any more, because he was either a human, died of natural death, or a zombie, killed by susceptible people.

In the model we considered there is no possibility for a zombie to become human again: once someone is infected, he can either survive, keeping infecting other humans, or be killed by susceptible people.

Figure 1: The three possible *states* in the *HZR* model.

3.1 Hypotheses

We will assume the following:

Discrete time. *The unit of time is 1 day.*

Uniqueness of the state. *An individual may assume only one state during each day.*

Thus, during a day each individual is not allowed to change his state: he has to maintain it till the end of the day. He may alter it only when passing from a day to the following.

3.2 Parameters

We considered some parameters to regulate the transitions from a state to the others (Figure 2).

- **Birth rate** α of humans.
- **Natural death rate** μ of humans.
- **Infection rate** β, probability that a zombie:
  1. **meets** a human;
  2. **infects** that human.
- **Removal rate** κ of zombies by humans.

We should also note that, assuming only the two hypotheses in §3.1, the total number of individuals (the *population*) can vary in time: when $\alpha \neq 0$, in fact, new individuals are born every day.

3.3 Equations

Once defined the states, some hypotheses and the parameters of the evolution, we are able to write the (recursive) equations modelling the number of susceptible people H_n and zombies Z_n in time. We choose not to study the dynamics of removed people R_n, because we judged it not interesting for our analysis.

Given a day n of the evolution, the following one $n+1$:

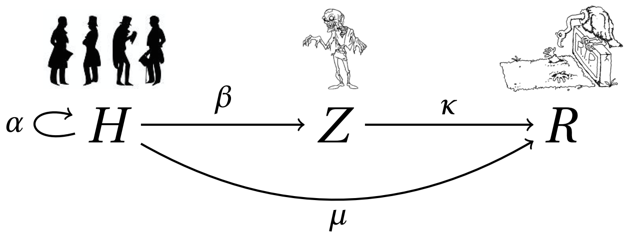

Figure 2: Parameters in the *HZR* model.

- a certain number αH_n of humans are born;
- a certain number μH_n of humans die;
- some susceptible people are infected and become zombies. Actually, each zombie infects βH_n humans during the day n: so, all the zombies Z_n infect $\beta H_n Z_n$ humans;
- in a similar way, humans are always trying to defeat zombies, and some of them die. Each susceptible kills κZ_n zombies during the day n, thus all the susceptible people H_n kill $\kappa H_n Z_n$ zombies.

Therefore, we can write the following recursive equations:

$$\begin{cases} H_{n+1} = H_n + \alpha H_n - \mu H_n - \beta H_n Z_n \\ Z_{n+1} = Z_n + \beta H_n Z_n - \kappa H_n Z_n \end{cases} \tag{1}$$

4 A simplified *HZR* model

At this point, we started studying empirically the behaviour of the functions H_n and Z_n, and we noticed that — for the short periods we initially considered, about 1 or 2 weeks — the influence of the births and natural deaths was nearly insignificant for our study, because generally they are very small rates. In addition, the α and μ parameters made our model more complicated and harder to analyse.

Thus we decided to study a simpler case, with $\alpha = 0$ and $\mu = 0$, leading us to add a third assumption among our hypotheses:

Closed system. *The total number of individuals is invariable in time.*

When talking about the *total number of individuals*, we mean $H_n + Z_n + R_n$: so, also removed people belong to the population.

With this assumption we can simplify the equations in (1), which become:

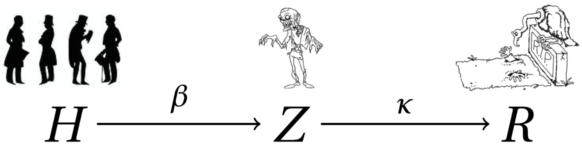

Figure 3: Parameters in the simplified *HZR* model.

$$\begin{cases} H_{n+1} = H_n - \beta H_n Z_n \\ Z_{n+1} = Z_n + \beta H_n Z_n - \kappa H_n Z_n \end{cases} \tag{2}$$

5 Some preliminary simulations

In order to get into the nature of the system, it's worth making some simulations before diving into any proofs. We set the parameters to some values we supposed to be quite realistic and plotted the resulting graphs for H_n and Z_n, using the software *Wolfram Mathematica*.

The initial population we always referred to is $H_0 = 10^6$, which could belong to a big city (or a small country). We chose to make the disease start from a small number of zombies $Z_0 = 10$. We discovered that β and κ had to be quite close each to the other to get non-banal cases (where zombies overcame humans — or *vice versa* — only after one or few days!): thus we set them to $\beta = 10^{-5}$ and $\kappa = 9.999 \cdot 10^{-6}$ (Figure 4).

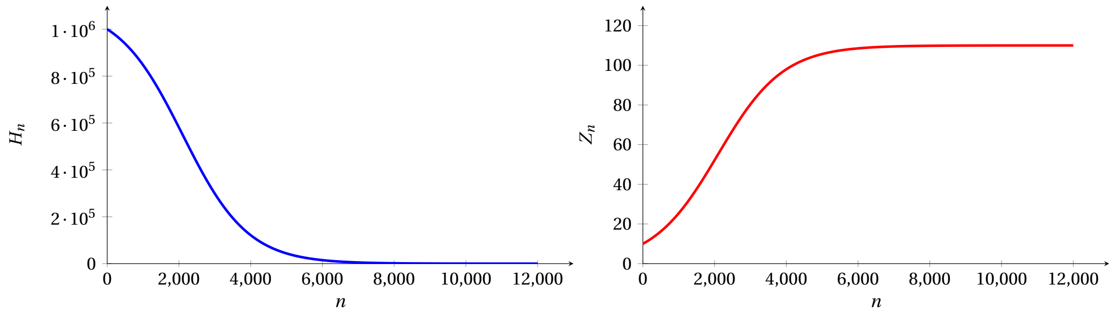

Figure 4: A sample simulation.

In fact, changing a little bit the parameters β and κ heavily affects the evolution, in terms of **maximum number of zombies** and of **time** needed to get to a state of equilibrium, as shown respectively in Figures 5 and 6.

Since we noticed that κ and β had to be quite near each other, we introduced a new parameter c, which combines both the parameters of the evolution:

$$c = \frac{\kappa}{\beta} \tag{3}$$

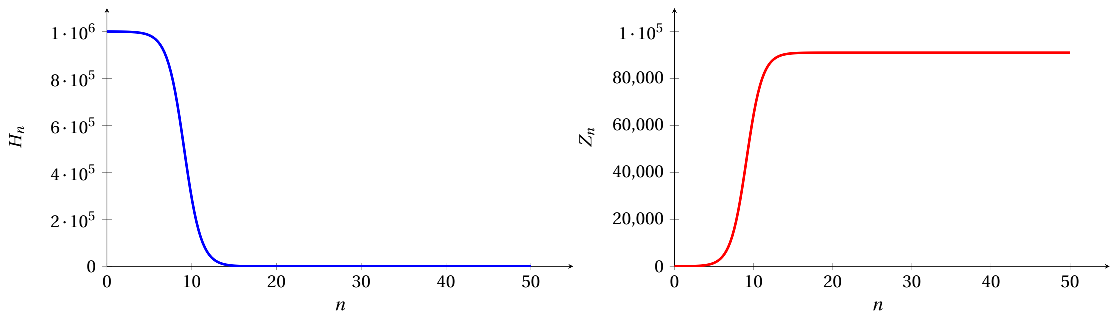

Figure 5: Changing β to $1.1\cdot10^{-5}$ affects the behaviour of the system.

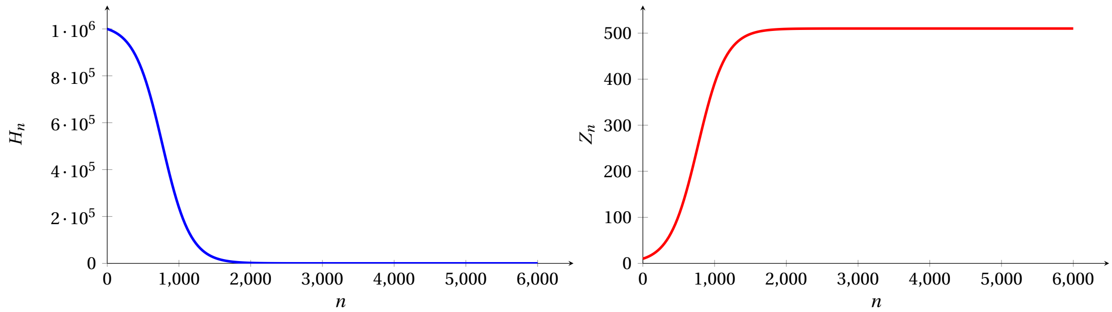

Figure 6: Changing κ to $9.995\cdot10^{-6}$ has an heavy influence too.

Therefore, we should choose a value for c very near to 1: in Figure 4 we have $c = 0.9999$, in Figure 5 $c = 0.909$, in Figure 6 $c = 0.9995$.

5.1 Making predictions

At this point, we empirically noticed that there is a certain maximum value for Z_n when the equilibrium is reached: we call this quantity Z_∞. But how do we calculate it?

Moreover, can we predict if zombies always win? Are there some cases for which humans manage to kill all the zombies?

We are not able to answer these questions only using the recursive equations in (2).

6 Looking for analytical equations

Although the found recursive equations (2) correctly describe the evolution of the system, they are quite **inefficient** for a simulation or for predictions. In order to gather some information about a late state of the system, we need to know *all* the previous states, whose computation takes uselessly long.

In fact, given the parameters and the initial conditions H_0 and Z_0, after only two days, we'll have a number of humans:

$$\begin{aligned} H_2 &= H_1 - \beta H_1 Z_1 \\ &= H_0 - \beta H_0 Z_0 - \beta \left(H_0 - \beta H_0 Z_0\right) \left[Z_0 + \left(\beta - \kappa\right) H_0 Z_0\right] \end{aligned}$$

Our next goal is to obtain the analytical equations for both H_n and Z_n, depending only on n and not on all the previous states of the evolution.

6.1 Invariants

To find out the analytical form of the functions H_n and Z_n, we will need to study some particular quantities involved in the evolution, called *invariants*.

Definition 1. *Let's call **invariant** of the model a quantity that is constant during the evolution.*

6.1.1 Total population

As stated in the assumption of *closed system* (see §4), we have that the total population

$$H_n + Z_n + R_n$$

is constant in time. Therefore, given the Definition 1, it is an invariant.

6.1.2 The ratios $\Delta Z_n / \Delta H_n$, $\Delta R_n / \Delta H_n$ and $\Delta Z_n / \Delta R_n$

In our simulations, we also found out that the ratios between the daily increments $\frac{\Delta Z_n}{\Delta H_n}$, $\frac{\Delta R_n}{\Delta H_n}$ and $\frac{\Delta Z_n}{\Delta R_n}$ should be invariants. We defined the *increment* as following.

Definition 2. *The **increment** of a quantity (e.g. H_n) is defined as:*

$$\Delta H_n = H_{n+1} - H_n$$

Therefore, we can rewrite the ratios using equations (2):

$$\frac{\Delta Z_n}{\Delta H_n} = \frac{\beta \cancel{H_n Z_n} - \kappa \cancel{H_n Z_n}}{-\beta \cancel{H_n Z_n}} = \frac{\kappa}{\beta} - 1 = c - 1 \tag{4a}$$

$$\frac{\Delta R_n}{\Delta H_n} = \frac{\kappa \cancel{H_n Z_n}}{-\beta \cancel{H_n Z_n}} = -\frac{\kappa}{\beta} = -c \tag{4b}$$

$$\frac{\Delta Z_n}{\Delta R_n} = \frac{\beta \cancel{H_n Z_n} - \kappa \cancel{H_n Z_n}}{\kappa \cancel{H_n Z_n}} = \frac{\beta}{\kappa} - 1 = \frac{1}{c} - 1 \tag{4c}$$

Since c is a constant parameter, we conclude that the studied ratios are invariants. ■

6.1.3 The invariant p

The ratio (4a) is quite interesting, because it involves both the quantities H_n and Z_n we are studying:

$$\frac{\Delta Z_n}{\Delta H_n} = c - 1$$
$$\Delta Z_n = (c-1)\Delta H_n$$
$$Z_{n+1} - Z_n = (c-1)\left(H_{n+1} - H_n\right)$$
$$Z_n + (1-c)H_n = Z_{n+1} + (1-c)H_{n+1}.$$

We have just obtained a new invariant:

$$p = Z_n + (1-c)H_n \tag{5}$$

which provides a useful relation between H_n and Z_n.

6.2 The search of a suitable model

Now that we have some more information about the quantities and the invariants involved, it is possible to find a suitable model. We found them studying two particular ratios.

6.2.1 The logistic equation

Let's consider the ratio between the increment ΔZ_n of zombies in a day n and the number Z_n of zombies in the same day:

$$\frac{\Delta Z_n}{Z_n} = \frac{\beta H_n \cancel{Z_n} - \kappa H_n \cancel{Z_n}}{\cancel{Z_n}} = (\beta - \kappa)H_n$$

If we replace κ with βc, using the equation (3), we obtain that

$$(\beta - \kappa)H_n = \beta(1-c)H_n$$

Keeping in mind equation (5), it is possible to replace $(1-c)H_n$ with $p - Z_n$. Thus:

$$\frac{\Delta Z_n}{Z_n} = \beta(p - Z_n)$$

which can be rearranged into

$$Z_{n+1} - Z_n = \beta\left(p - Z_n\right)Z_n$$

We found the following equation:

$$Z_{n+1} = (1+\beta p)Z_n - \beta Z_n^2 \tag{6}$$

It proves that Z_n satisfies a **logistic equation**, where βp is its *growth rate*. ■

In a similar way, we can work on the ratio between the increment ΔH_n of humans in a day n and the number H_n of humans in the same day:

$$\frac{\Delta H_n}{H_n} = \frac{-\beta \cancel{H_n} Z_n}{\cancel{H_n}} = -\beta \left[p - (1-c) H_n \right]$$

$$\implies H_{n+1} - H_n = -\beta H_n \left[p - (1-c) H_n \right]$$

$$H_{n+1} = (1 - \beta p) H_n + \beta (1-c) H_n^2 \tag{7}$$

This proves that H_n satisfies a logistic equation too, where $-\beta p$ is its *growth rate*. ■

6.3 The logistic function

We proved that both H_n and Z_n satisfy a logistic equation. Therefore, it is now possible to write the analytical form of their **logistic function**.

A generic logistic function f_t can be analytically written knowing three parameters:

- the **initial value** f_0;
- the **growth rate** r;
- the **upper value** L.

Thus, the generic function f_t has the form

$$f_n = \frac{L}{1 + \left(\frac{L}{f_0} - 1 \right) e^{-rn}}$$

6.3.1 Zombies' function

We already know two out of three parameters: the initial value is Z_0 (that is a given number) and the growth rate βp, from the logistic equation (6). Now we have to find the upper value Z_∞.

First, we notice that, when $n \to +\infty$,

$$Z_\infty = \lim_{n \to +\infty} Z_n = \lim_{n \to +\infty} Z_{n+1}$$

Therefore, we can replace Z_n and Z_{n+1} with Z_∞ in the logistic equation (6) when $n \to +\infty$:

$$Z_{n+1} = (\beta p + 1) Z_n - \beta Z_n^2$$

$$\implies Z_\infty = (\beta p + 1) Z_\infty - \beta Z_\infty^2$$

This is a second degree equation we can solve very easily. Excluding the banal solution $Z_\infty = 0$, we can divide for Z_∞:

$$1 = \beta p + 1 - \beta Z_\infty$$

$$Z_\infty = p \tag{8}$$

We have just found that the upper value Z_∞ is the invariant p from equation (5)! Now we can write the full logistic function Z_n describing the number of zombies in time n:

$$\boxed{Z_n = \frac{p}{1 + \left(\frac{p}{Z_0} - 1\right) e^{-\beta p n}}} \tag{9}$$

6.3.2 Humans' function

In a very similar way, we can obtain the humans' logistic function. We already know the initial value H_0 and the growth rate $-\beta p$ from the logistic equation (7). Also in this case we do not have the upper value H_∞.

We obtained it from p in equation (5):

$$p = Z_n + (1-c)H_n$$

In fact, when the humans win against the zombies, we have $Z_\infty = 0$ for $n \to +\infty$:

$$p = \cancel{Z_\infty} + (1-c)H_\infty \implies p = (1-c)H_\infty$$

$$H_\infty = \frac{p}{1-c} \tag{10}$$

It is now possible to write the full logistic function H_n, which describes the number of humans in time n:

$$\boxed{H_n = \frac{p}{1 - c + \left(\frac{p}{H_0} - 1 + c\right) e^{\beta p n}}} \tag{11}$$

It's worth noticing that the logistic functions in equations (9) and (11) are not the actual solutions of the logistic equations (6) and (7). They are only an excellent approximation, since we are working in a discrete time and with a discrete number of people (although it is very big).

7 Who wins?

During our simulations, using the software *Wolfram Mathematica*, we understood that the parameter c decides whether humans or zombies win.

In fact, the ratio c between the infection rate β and the removal rate κ is decisive to describe if *zombism* infects more people than the number of zombies killed by humans (or not).

To find the conditions on c, we studied first a particular case: when $c = 1 + \frac{Z_0}{H_0}$, nobody survives.

7.1 Nobody survives

We discovered that, in the case $c = 1 + \frac{Z_0}{H_0}$, both humans and zombies die for $n \to +\infty$. In this situation, we have that

$$p = Z_0 + \left[1 - \left(1 + \frac{Z_0}{H_0}\right) H_0\right] = Z_0 - \frac{Z_0}{\cancel{H_0}}\cancel{H_0} = 0$$

Simply replacing $p = 0$ in the logistic functions (9) and (11) is not possible: we would obtain an indeterminate form $\frac{0}{0}$. So, to find the functions H_n and Z_n, we have to calculate the limits:

$$\lim_{p\to 0} H_n \qquad \lim_{p\to 0} Z_n$$

Let's see the zombies' limit first:

$$\begin{aligned}
\lim_{p\to 0} Z_n &= \lim_{p\to 0} \frac{p}{1 + \left(\dfrac{p}{Z_0} - 1\right) e^{-\beta p n}} = \\
&= \lim_{p\to 0} \frac{p}{1 + \dfrac{p - Z_0}{Z_0} e^{-\beta p n}} = \\
&= \lim_{p\to 0} \frac{p Z_0}{Z_0 + (p - Z_0) e^{-\beta p n}} = \frac{0}{0}
\end{aligned}$$

To eliminate the indeterminate form, let's apply De L'Hôpital's theorem:

$$\lim_{p\to 0} \frac{p Z_0}{Z_0 + (p - Z_0) e^{-\beta p n}} \overset{\text{H}}{=} \lim_{p\to 0} \frac{Z_0}{\left[1 - \beta n \left(p - Z_0\right)\right] e^{-\beta p n}} = \frac{Z_0}{1 + Z_0 \beta n} \tag{12}$$

In a similar way, we calculated the limit of H_n for $p \to 0$:

$$\begin{aligned}
\lim_{p\to 0} H_n &= \lim_{p\to 0} \frac{p}{1 - c + \left(\dfrac{p}{H_0} - 1 + c\right) e^{\beta p n}} = \\
&= \lim_{p\to 0} \frac{p H_0}{p + Z_0 \left(e^{\beta p n} - 1\right)} = \frac{0}{0}
\end{aligned}$$

Applying De L'Hôpital's theorem:

$$\lim_{p\to 0} \frac{pH_0}{p+Z_0\left(e^{\beta pn}-1\right)} \overset{\text{H}}{=} \lim_{p\to 0} \frac{H_0}{1+Z_0\beta n e^{\beta pn}} = \frac{H_0}{1+Z_0\beta n} \tag{13}$$

In conclusion, from equations (12) and (13) we have that:

$$\begin{cases} H_n = \dfrac{H_0}{1+Z_0\beta n} \\ Z_n = \dfrac{Z_0}{1+Z_0\beta n} \end{cases} \tag{14}$$

that are two homographic functions, as we can see in the graphs of Figure 7.

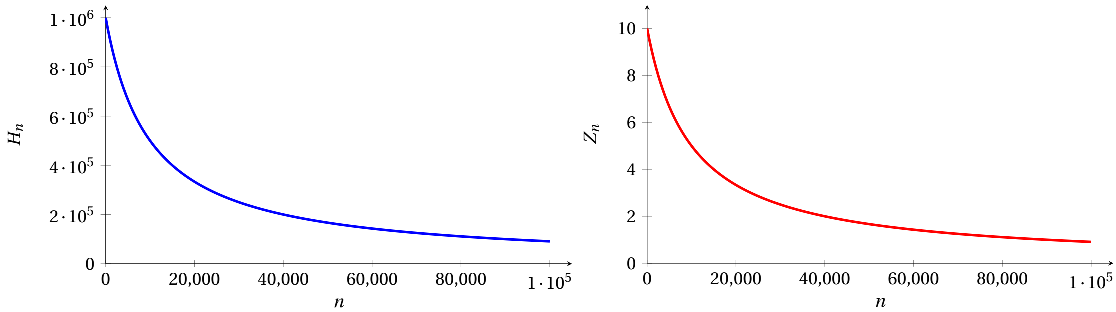

Figure 7: For $c = 1.00001$, nobody wins.

7.2 Zombies win

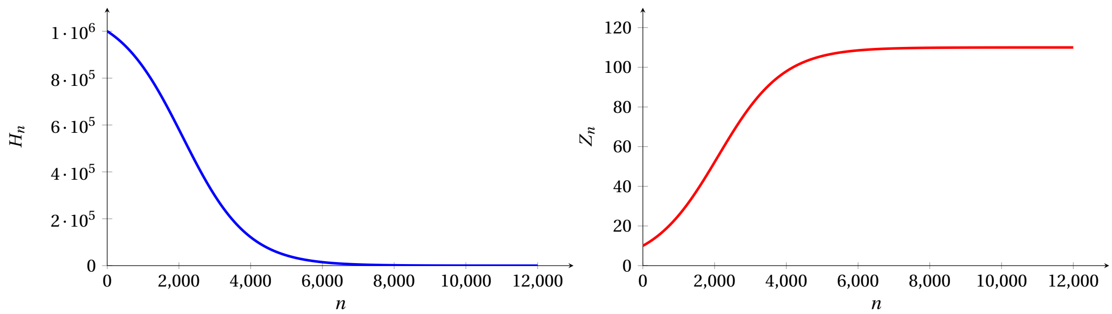

Figure 8: For $c = 0.9999$, zombies win.

We obtained through the simulations that, when $c < 1 + \frac{Z_0}{H_0}$, zombies always win and no human remains. In Figure 8, for example, $c = 0.9999$.

7.2.1 A particular case: $c = 1$

When $c = 1$, we noticed that the zombies are constant in time. In fact, from equation (5), $p = Z_0 + \cancel{(1-c)H_0} = Z_0$. Thus, the logistic functions Z_n and H_n from equations (9) and (11) become respectively:

$$\begin{cases} H_n = \dfrac{Z_0}{\cancel{1-c} + \left(\dfrac{Z_0}{H_0} \cancel{-1+c}\right) e^{rn}} \\ Z_n = \dfrac{Z_0}{1 + \left(\dfrac{\cancel{Z_0}}{\cancel{Z_0}} - 1\right) e^{-rn}} \end{cases} \implies \begin{cases} H_n = H_0 e^{-rt} \\ Z_n = Z_0 \end{cases} \tag{15}$$

that describes a simple exponential decrease for the humans H_n. On the other hand, zombies Z_n remain constant (equal to the initial value Z_0), as we can see from Figure 9.

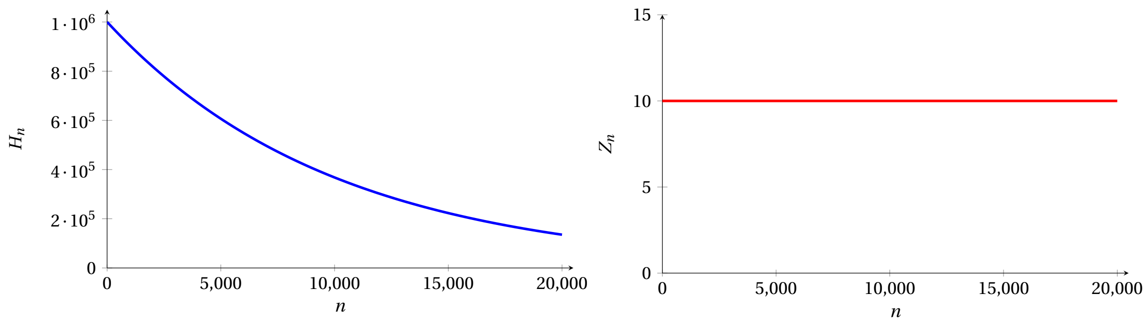

Figure 9: For $c = 1$, zombies win and are constant in time.

7.3 Humans win

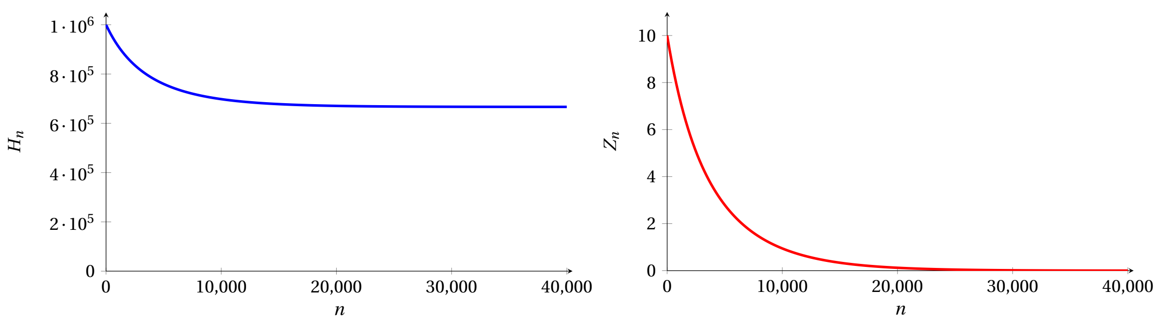

Figure 10: For $c = 1.00003$, humans win.

Instead, when $c > 1 + \frac{Z_0}{H_0}$, we discovered that humans win, because they manage to kill all the zombies. In Figure 10, for example, $c = 1.00003$.

8 Conclusion

We have produced a model describing a Zombie outbreak, consistently with the rules we observed in movies and literature.

We were interested in understanding what the model predicts, so we ran our model with simulations to verify if the dynamics of the model is qualitatively the same of an apocalypse such as it is portrayed by popular renditions.

In order to facilitate our analysis we made a model that uses only a few parameters.

We estimated the populations in the movies to fit the parameters of the model. In the series Walking Dead the story elapsed over a time of about 2 years, the zombie population overtook humans within just a few weeks, leaving only a small fraction of the initial population. The surviving humans killed all the zombies in 2 years.

We have explored the possible dynamics through a large number of simulations, observing the behaviour of the model and searching for parameters that show a certain plausibility with the stories of the TV series *The Walking Dead*.

We have found that all our results depend on poor assumptions of the model.

Our analysis shows that there is no stable state in which zombie and human could coexist. We have found the values of the parameters for which the entire human population is eliminated. Another scenario is that all zombies are killed and human population is strongly reduced.

The relative values of κ and β have a great importance: in fact, to get results that are consistent with the TV series we referred to, they had to be quite close each other. Otherwise, after only one or two days all humans or zombies could have died!

The two parameters are crucial for determining whether humans can survive a Zombie outbreak, and we found the condition for humans' victory using the analytical function that we managed to obtain.