EGYPTIAN FRACTIONS

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Presentation of the research topic:

This year the Math.en.jeans workshop proposed to work with Egyptian fractions. Egyptians used only fractions like $\frac{1}{n}$ *such that* $n \in \mathbb{N}_0$ *and did not know negative fractions. These fractions will be called unitary fractions or unit fractions. We want to write proper irreducible fractions* $\frac{a}{b}$*, such that* a *and* b *are non-zero natural numbers, as a sum of distinct unit fractions. This sum is called Egyptian fraction.*

Brief presentation of the conjectures and results obtained:

The aims of this article are to verify if we can always write a proper irreducible fraction $\frac{a}{b}$ as an Egyptian fraction; to verify if there are different and eventually infinite possible expansions; to explore different ways to expand a proper fraction, comparing various methods in order to understand if there is a preferable one, depending on the results they lead to.

We studied Fibonacci's method, Golomb's method and a method based on practical numbers, retracing the original proofs, introducing new results and proposing some variants to the methods. Most importantly, we observed that through Fibonacci's algorithm every proper fraction can be expanded into Egyptian fractions, and the ways to do that are infinite in number.

We proposed a new original method based on a geometric approach to the problem.

We studied the tree composed of the unit fractions that expand a given proper fraction, designing a function that allows to determine the terms of the tree. Thanks to the tree we can also expand natural numbers and improper fractions.

Introduction

Egyptians used fractions as $\frac{1}{n}$ with $n \in \mathbb{N}_0$ (we denote the set of non-zero natural numbers by $\mathbb{N}_0$). These fractions will be called "**unit fractions**" (U.F.). Instead of proper fractions, Egyptians used to write them as a sum of distinct U.F. So every time they wanted to express a fractional quantity, they used a sum of U.F., each of them different from the others in the sum. This expansion of a proper fraction is called "Egyptian fraction".

For example, the fraction $\frac{3}{4}$ as an Egyptian fraction will not be $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$, but will be $\frac{1}{2}+\frac{1}{4}$.

The first question someone could arise is: why did Egyptians use this particular method? Even if there are many hypothesis, none of them is ascertained. The most accredited of them is that they used U.F. to simplify practical suddivisions. Mathematically, it is easier for us to think in terms proper fractions, but on a concrete level, it is simpler to operate with unitary parts of a whole. On a document named "Rhind Papyrus" of Ahmes (one of the oldest known mathematical manuscripts, dating from around 1650 B.C.) are listed in a table the expansions of every proper fraction with 2 as numerator and from 3 to 101 as denominator. This is one of the most important examples of Egyptian mathematics and represents the base of Egyptian fractions.

There are various explanations as to why the Egyptians chose to use such representations but perhaps the most compelling is the one given in the book "The Man Who Loved Only Numbers" by the legendary mathematician André Weil:

He thought for a moment and then said: It is easy to explain. They took a wrong turn! [11]

Approaching the subject some questions came up:

Can we write every proper fraction as a sum of unit fractions?

Are there different methods to decompose a proper fraction?

Are there different types of decomposition?

Are there infinite expansions for each proper fraction?

Is there a best expansion? If so, would it be the shortest one or the one that leads to a sum with the lowest denominators?

In the following chapters we will give some answers to these questions.

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Chapter 1

Fibonacci's Method

The first way we propose was given by Fibonacci, also known as Leonardo Pisano, in 1202. His method is to apply the **greedy algorithm** which consists in subtracting the largest possible U.F. from the given fraction so that the result is non-negative. We have to repeat this on the remainder until we find a fraction that is itself a U.F. not equal to one already written down.

1.1 The Greedy method.

We can note that:

Theorem 1.1.1. *Subtracting the largest possible unit fraction that keeps a non negative difference from a proper irreducible fraction* $\frac{a}{b}$*, that is, such that* $\frac{a}{b} - \frac{1}{k} \geq 0$*, the resulting fraction numerator is strictly less than the one of the initial fraction and the resulting fraction denominator is strictly greater than the one of the initial fraction.*

Proof. Chosen a non-zero natural number k, it must also be different from one. Indeed, if we subtract from the proper fraction $\frac{a}{b}$ the U.F. $\frac{1}{1}$ (so with $k = 1$), the result will be negative. If $k \geq 2$ the difference between a proper fraction and a U.F. is:

$$\frac{a}{b} - \frac{1}{k} = \frac{ka - b}{bk} \qquad \Rightarrow \qquad \frac{a}{b} = \frac{1}{k} + \frac{ka - b}{bk}$$

Since $\frac{1}{k}$ is the largest unit fraction that we can subtract from $\frac{a}{b}$, then

$$\frac{1}{k-1} > \frac{a}{b}$$

then multiplying both sides by b we have

$$\frac{b}{k-1} > a$$

and now

$$b > a(k-1) \qquad \Rightarrow \qquad ka - b < a$$

Since $k \geq 2$ the denominators of the fractions must be strictly increasing. □

Theorem 1.1.2. *Given an irreducible proper fraction* $\frac{a}{b}$*, the Fibonacci algorithm produces an expansion with at most a number of distinct unit fractions equal to the numerator (a).*

Proof. The previous theorem states that $ka-b < a$, $ka-b > 0$ and $ka-b \in \mathbb{N}$. Then the numerator strictly decreases after each step therefore the number of terms in the representation of $\frac{a}{b}$ is at most a.

At last, the only way this method could go wrong would be if two of the fractions were equal (in fact this is not allowed in Egyptian fractions). But this can't happen because if we had two successive terms $\frac{1}{n}$ and $\frac{1}{m}$ with $n = m$, we could have chosen $\frac{1}{n-1}$ instead of $\frac{1}{n}$.

Infact $\frac{1}{n} + \frac{1}{n} > \frac{1}{n-1}$ since $n > 2$. □

Answer: Fibonacci's Method guarantees that every proper fraction can be expanded into an infinite number of distinct unit fractions.

1.2 The value of k

Operatively we can use this result:

Theorem 1.2.1. *Given a irreducible proper fraction* $\frac{a}{b}$*, holds*

$$\frac{a}{b} = \frac{1}{k} + \frac{A}{B} \qquad \textit{with} \qquad k = \left\lceil \frac{b}{a} \right\rceil \qquad \textit{and} \qquad A < a$$

.

Proof. Assuming that:

$$\frac{a}{b} - \frac{1}{k} > 0 \Rightarrow \frac{1}{k} < \frac{a}{b} \Rightarrow k > \frac{b}{a}$$

where k is the lowest denominator before $\frac{a}{b} - \frac{1}{k}$ is less than 0.

From the theorem 1.1.1 holds $A < a$ □

Example: Let's consider a proper fraction $\frac{a}{b}$ where a is 791 and b is 3517.

So, if we want to find k, we have to divide 3517 for 791. In this case the result of $\frac{3517}{791}$ is $4,4462705...$ So, k will be 5 because it is $\left\lceil \frac{b}{a} \right\rceil$. In symbols: $\frac{a}{b} - \frac{1}{k} > 0 \Rightarrow \frac{791}{3517} - \frac{1}{5} > 0$.

Therefore

$$\frac{791}{3517} = \frac{1}{5} + \frac{384}{17855} \qquad \text{where} \qquad 384 < 791.$$

1.3 Number of unit fractions composing the sum

Fibonacci observed that it is possible to expand an irreducible proper fraction into a finite number of unit fractions exactly by at most a number of addends equal to the value of initial numerator.

Now that we have established a way (of course that is not the only one) to define k, we are going to study the problem of searching the minimal number of U.F. the initial one can be expanded by.

We will start with the lowest number of U.F., which is, of course, 2 (we are not going to consider 1 because it refers to a proper fraction which already is unit). As a consequence we can say that:

$$\frac{a}{b} - \frac{1}{k} = \frac{1}{p}.$$

As you can see, we have introduced a new variable, p.

We assume that $p \in \mathbb{N}_0$ and now we have to find p using the other variables. Starting from the previous equation, we come to:

$$\frac{ak-b}{bk} = \frac{1}{p} \qquad \Rightarrow \qquad p = \frac{bk}{ak-b}$$

so we can easily find the value of p if it is a natural number.

But this is not the only Egyptian fraction that represents the given proper fraction. In fact, there are infinite ways to come to that proper fraction summing U.F. Indeed, we can break down not only fractions with numerator greater than 1, but we can expand unit fractions, too.

Theorem 1.3.1. *For all $x \in \mathbb{N}_0$ holds:*

$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$$

Proof. If we have the difference: $\frac{1}{x} - \frac{1}{y}$ where x and y are non-zero natural numbers, the result will be $\frac{y-x}{xy}$. To guarantee that this difference is a U.F. we impose that $y - x = 1$, so that $y = x + 1$. As a consequence we can say that subtracting to a U.F. another U.F. which is strictly less than it, the result will be a U.F., too. The formula is:

$$\frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} \implies \frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$$

□

Thanks to this, we can prove that a proper fraction can be expanded into a sum of unit fractions in an infinite number of ways.

Example: Chosen a proper fraction $\frac{3}{5}$, using the Fibonacci's method, we have to calculate the value of $k = \left\lceil \frac{5}{3} \right\rceil = 2$. By subtracting from the proper fraction the U.F. $\frac{1}{k}$ we come to:

$$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$$

But now, thanks to the identity above, we can expand every single addend into other unit fractions, and so on.

Example:

$$\frac{1}{2} = \frac{1}{2+1} + \frac{1}{2 \cdot (2+1)} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{10} = \frac{1}{10+1} + \frac{1}{10 \cdot (10+1)} = \frac{1}{11} + \frac{1}{110}$$

So $\frac{3}{5} = \frac{1}{3} + \frac{1}{6} + \frac{1}{11} + \frac{1}{110}$.

The requirement of distinct fractions does not increase the number of fractions. In fact, for n odd we can use the identity:

$$\frac{1}{n} + \frac{1}{n} = \frac{2}{n+1} + \frac{2}{n(n+1)}$$

that shows that the sum of two equal unit fractions with odd denominator can be written as the sum of two distinct unit fractions.

For n even instead the sum of two equal unit fractions with even denominator can be written as a single unit fraction.

$$\frac{1}{n} + \frac{1}{n} = \frac{2}{n}$$

that shows that the sum of two equal unit fraction with even denominator can be written as a single unit fraction.

Now we can expand every U.F. into other unit fractions.

Answer: There are infinite sums of distinct unit fractions that expand an irreducible proper fraction.

1.4 Merits and limits

Fibonacci found an algorithm which is based on the greedy procedure as we have already said, in which the largest possible unit fraction is chosen for each term of the expansion.

Let's make an example, such as $\frac{3}{7}$. Its greedy expansion will be:

$$\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$$

It can be also written as

$$\frac{3}{7} = \frac{1}{6} + \frac{1}{7} + \frac{1}{14} + \frac{1}{21}$$

As we can see, this second expansion has more terms, but the highest denominator is lower than the Fibonacci's one (231 vs 21).

Alternatively, we can write $\frac{3}{7}$ as

$$\frac{3}{7} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28}$$

that has three terms, like the greedy expansion, and 28 is the highest denominator, which is lower than 231.

To sum up, Fibonacci's algorithm is not the best method to find an expansion that generates the lowest denominators.

Now we want to discover if the algorithm does or does not lead to the shortest expansion.

Chosen a fraction $\frac{4}{49}$, by the greedy procedure it would be expanded into:

$$\frac{4}{49} = \frac{1}{13} + \frac{1}{213} + \frac{1}{67841} + \frac{1}{9204734721}$$

but it can also be written as:

$$\frac{4}{49} = \frac{1}{14} + \frac{1}{98}$$

so we have easily proved that the Fibonacci's algorithm does not lead to the shortest expansion.

Therefore the merits of this method are:

- It works with all proper fractions;
- It can break every proper fraction into an Egyptian fraction.

But its limits are:

- The denominators of the resulting fractions can grow quite big.
- It's not always the shortest expansion.

Chapter 2

Golomb's method

There is an algorithm due to American mathematician Solomon Wolf Golomb that we can use in order to expand an irreducible proper fraction into unit fractions.

Given a positive proper fraction $\frac{a}{b}$, exist two non-zero natural number A and B such that $aA = bB + 1$; then:

$$\frac{a}{b} = \frac{aA}{bA} = \frac{bB}{bA} + \frac{1}{bA} = \frac{B}{A} + \frac{1}{bA}$$

Since $aA = bB + 1$ and $a < b$ then $aA > bB > aB$, so $0 < \frac{B}{A} < 1$. We can apply the above procedure for $\frac{B}{A}$ (we can suppose that $gcd(A, B) = 1$).

On the other hand, we have $aA > bB > AB$, hence $B < a$, which guarantees the finiteness of the method.

Example:

Given $a = 3, b = 7$ such that $\frac{a}{b} = \frac{3}{7}$. We can find that $3A = 7B + 1$ with $A = 5, B = 2$. So

$$\frac{a}{b} = \frac{1}{bA} + \frac{B}{A} \qquad \Rightarrow \qquad \frac{3}{7} = \frac{1}{35} + \frac{2}{5}$$

Now we have to carry on with the fraction $\frac{B}{A} = \frac{2}{5}$. So, we can find that $2C = 5D + 1$ with $C = 3, D = 1$. So

$$\frac{B}{A} = \frac{1}{AC} + \frac{D}{C} \qquad \Rightarrow \qquad \frac{2}{5} = \frac{1}{15} + \frac{1}{3}$$

Therefore

$$\frac{a}{b} = \frac{1}{bA} + \frac{B}{A} = \frac{1}{bA} + \frac{1}{AC} + \frac{D}{C} \qquad \Rightarrow \qquad \frac{3}{7} = \frac{1}{35} + \frac{2}{5} = \frac{1}{35} + \frac{1}{15} + \frac{1}{3}$$

Now the problem is how to find the numbers A and B. The method that we will use is based on Bezout's theorem:

Theorem 2.0.1. *(Bezout). For non-zero a and b in $\mathbb{Z}$, there are x and y in $\mathbb{Z}$, such that*

$$gcd(a, b) = ax + by$$

In particular, when a and b are relatively prime, there are x and y in $\mathbb{Z}$ such that $ax + by = 1$.

The Equation $gcd(a,b) = ax + by$ is called **Bezout's identity**.

Since we can suppose that $\frac{a}{b}$ is a fraction in lowest terms then $gcd(a,b) = 1$. Therefore there are A and B in $\mathbb{Z}$ such that $aA + bB = 1$. Whereas $a, b > 0$ necessarily A and B can't be both positive. We would like $B < 0$, in fact for the Golomb's method we use the theorem:

Theorem 2.0.2. *Let $a < b$ be positive integers with $a \neq 1$ and with $gcd(a,b) = 1$, and consider the fraction $0 < \frac{a}{b} < 1$ then there exist a natural number $0 < A < b$ and a natural number B such that*

$$aA = bB + 1$$

Proof. Bezout's theorem says that there exist x and y in $\mathbb{Z}$ such that $gcd(a,b) = ax + by$, but does not say that they are unique. It is possible to prove that exist infinite couples of integers that satisfy the Bezout's identity.

In fact, if x and y is one pair of Bezout's coefficients then also all pairs

$$\left(x - k\frac{b}{\gcd(a,b)},\ y + k\frac{a}{\gcd(a,b)}\right), \qquad \text{with} \qquad k \in \mathbb{Z}$$

satisfy the same identity.

In fact if (x, y) is a solution pair of the equation and $d = \gcd(a,b)$ then

$$a\left(x - k\frac{b}{d}\right) + b\left(y + k\frac{a}{d}\right) = ax - ak\frac{b}{d} + by + bk\frac{a}{d} = ax + by = d.$$

We would like $A > 0$, $B < 0$ and $A < b$. But if we do not find such numbers we can choose an integer number k such that makes $A - kb > 0$, $B + ka < 0$ and $A - kb < b$. We want to prove that such number exist. Let $a, b \in \mathbb{N}$ with $a < b$ and $A, B \in \mathbb{Z}$ such that $aA + bB = 1$ and $A \cdot B < 0$, we can find $A' = A - kb$ and $B' = B + ka$ such that $A' > 0$, $B' < 0$ and $A' < b$.

We have to take $k \in \mathbb{Z}$ such that $\frac{A}{b} - 1 < k < -\frac{B}{a}$.

In fact if $k < -\frac{B}{a}$ then

$$A' = A - kb > A + \frac{B}{a}b = \frac{aA + bB}{a} = \frac{1}{a} > 0$$

and

$$B' = B + ka < B - \frac{B}{a}a = B - B = 0$$

and if $k > \frac{A}{b} - 1$ then

$$A' = A - kb < A + b\left(1 - \frac{A}{b}\right) = A + b - A = b$$

The last question is:

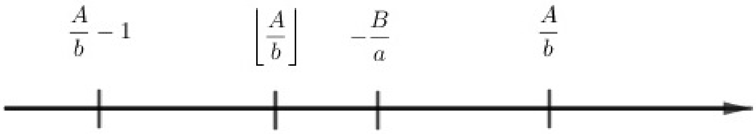
Is there such an integer number?

Let $a, b \in \mathbb{N}$ with $a < b$ and $A, B \in \mathbb{Z}$ such that $aA + bB = 1$ and $A \cdot B < 0$,

$$\exists k \in \mathbb{Z} : \frac{A}{b} - 1 < k < -\frac{B}{a}$$

.

A number with these properties exists and it is $k = \left\lfloor \frac{A}{b} \right\rfloor$



where $\lfloor x \rfloor$ is the floor function that takes a real number x as input and gives as output the greatest integer that is less than or equal to x.

It is clear from the definitions that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, then $\frac{A}{b} - 1 < \left\lfloor \frac{A}{b} \right\rfloor$.

But the following inequality holds too: $\left\lfloor \frac{A}{b} \right\rfloor < -\frac{B}{a}$.

In fact, $\frac{A}{b} > \left\lfloor \frac{A}{b} \right\rfloor \Rightarrow A > b\left\lfloor \frac{A}{b} \right\rfloor$. Since A and $b\left\lfloor \frac{A}{b} \right\rfloor$ are integer numbers, then $A \geq b\left\lfloor \frac{A}{b} \right\rfloor + 1$ then

$$\frac{A}{b} - \left\lfloor \frac{A}{b} \right\rfloor \geq \frac{b\left\lfloor \frac{A}{b} \right\rfloor + 1}{b} - \left\lfloor \frac{A}{b} \right\rfloor = \left\lfloor \frac{A}{b} \right\rfloor + \frac{1}{b} - \left\lfloor \frac{A}{b} \right\rfloor = \frac{1}{b} > \frac{1}{ab} = \frac{aA + bB}{ab} = \frac{A}{b} + \frac{B}{a}$$

Therefore

$$\frac{A}{b} - \left\lfloor \frac{A}{b} \right\rfloor > \frac{A}{b} + \frac{B}{a} \Rightarrow \left\lfloor \frac{A}{b} \right\rfloor < -\frac{B}{a}$$

. □

Hence $\frac{a}{b} = \frac{B}{A} + \frac{1}{bA}$ with $0 < A < b$ then $A \leq b - 1$ and so $bA \leq b(b-1)$.

We can repeat the procedure for every following step, in fact at each i-th step $A_i < A_{i-1}$ and so $A_i \cdot A_{i-1} \leq b(b-1)$.

Then the algorithm gives distinct unit fractions which have denominators at most $b(b-1)$.

The problem now is how to find an algorithm that finds the numbers A and B. Such algorithm is the Euclidean algorithm.

2.1 Euclidean algorithm

A constructive method for obtaining the Bezout's identity terms is to use the extended Euclidean algorithm. The Euclidean algorithm, or Euclid's algorithm, is an efficient method for computing the greatest common divisor (gcd) of two numbers, the largest number that divides both of them without leaving a remainder.

Theorem 2.1.1. *Given two non-zero natural numbers a and b with $a < b$, we divide $\frac{b}{a}$ and assign the remainder of the division to r. If r = 0 then a= gcd(a,b) otherwise we assign a = b and b = r and repeat the division again.*

The Euclidean algorithm is basically a continual repetition of the division algorithm for integers. The point is to repeatedly divide the divisor by the remainder until we get 0. The gcd is the last non-zero remainder in this algorithm. Keeping in mind the quotients obtained during the algorithm development, two integers p and q can be determined such that ap + bq = gcd(a, b).

2.2 Method for Egyptian fractions

Let's consider a positive proper fraction $\frac{a}{b}$:

(1) Find A e B of the Bezout's Identity using the Euclidean algorithm then $aA + bB = 1$;

(2) If $0 < A < b$ and $B < 0$ then go to the next step;

else we calculate $A' = A - kb$ and $B' = B + ka$ with $k = \left\lfloor \frac{A}{b} \right\rfloor$ and, for the theorem 2.0.2 with get $0 < A' < b$ and $B' < 0$ and now go to the next step;

(3) Use the Golomb's method

Example:
Let's consider the fraction $\frac{a}{b} = \frac{38}{121}$

(1) We find two numbers A and B with the Euclidean algorithm:

$121 = 38 \cdot 3 + 7$

$38 = 7 \cdot 5 + 3$

$7 = 3 \cdot 2 + 1$

$3 = 1 \cdot 3 + 0$

Let's start with the second-last identity: we explicit 1 and we substitute in it the number 3 made explicit in the immediately preceding equation.

$1 = 7 - 3 \cdot 2$

$1 = 7 - (38 - 7 \cdot 5) \cdot 2$

We group the common factors and we continue to substitute the rest of the previous equation (proceeding from bottom to top) until an expression is obtained in the numbers A and B.

$1 = (121 - 38 \cdot 3) - \{[38 - (121 - 38 \cdot 3) \cdot 5] \cdot 2\}$

$1 = 121 \cdot 11 + 38 \cdot (-35)$

At the end $38A + 121B = 1 \quad \Rightarrow \quad A = -35, B = 11$

(2) However A and B do not respect the desired conditions then we have to calculate

$$k = \left\lfloor \frac{A}{b} \right\rfloor = \left\lfloor \frac{-35}{121} \right\rfloor = \lfloor -0.2 \rfloor = -1$$

Therefore

$$A' = A - kb = 86 \qquad \text{and} \qquad B' = B + ka = -27 \qquad \Rightarrow \qquad A' > 0, B' < 0, A' < b$$

(3) Now $38 \cdot 86 + (-27) \cdot 121 = 1$ (Bezout's identity) then $38 \cdot 86 = 27 \cdot 121 + 1$ and we can use the Golomb's method:

$$\frac{38}{121} = \frac{38 \cdot 86}{121 \cdot 86} = \frac{27 \cdot 121 + 1}{121 \cdot 86} = \frac{27}{86} + \frac{1}{121 \cdot 86} = \frac{27}{86} + \frac{1}{10406}$$

We have to repeat the same procedure until the proper fraction $\frac{a}{b}$ will be expanded as the sum of unit fractions, obtaining an Egyptian fraction.

2.3 Observations

The Golomb's algorithm is better than the Fibonacci one because the denominators are guaranteed to be less than $b(b-1)$.

The Golomb's denominators are not always greater than the Fibonacci's one, but while using Golomb's method there is a bound for the denominators, using the Fibonacci's method they can grow quite big.

Chapter 3

Method of practical numbers

In order to create a unit fraction expansion of a proper irreducible fraction $\frac{a}{b}$ we noticed that if a can be written as the sum of divisors of b, then we can easily reach our goal and, in particular, $\frac{a}{b}$ can be expanded with denominator always less than b itself.

Therefore to expand a given fraction with the numerator that can be expressed as a sum of denominator divisors as sum of Egyptian fractions, we express $\frac{a}{b}$ as sum of fractions with numerator a divisor of b over b.

For example:

$$\frac{a}{b} = \frac{7}{12} \qquad D_{12} = \{1, 2, 3, 4, 6, 12\}$$

And so $\frac{7}{12}$ can be expressed as sum of:

$$\frac{7}{12} = \frac{1}{12} + \frac{2}{12} + \frac{4}{12}; \qquad \frac{7}{12} = \frac{3}{12} + \frac{4}{12}; \qquad \frac{7}{12} = \frac{1}{12} + \frac{6}{12}$$

And simplifying all the equations we obtain sums of Egyptian fraction:

$$\frac{7}{12} = \frac{1}{12} + \frac{1}{6} + \frac{1}{3}; \qquad \frac{7}{12} = \frac{1}{4} + \frac{1}{3}; \qquad \frac{7}{12} = \frac{1}{12} + \frac{1}{2}$$

If the denominator of an irreducible fraction proper is a *practical number* then this technique allows to expand all the proper fractions with that denominator.

3.1 Practical numbers

Definition 3.1.1. *A practical number is a natural number N such that for all $n < N$, $n \in \mathbb{N}_0$ then n can be written as the sum of distinct divisors of N.*

Hence we thought to divide the set of natural numbers $\mathbb{N}$ into two subsets: the set of practical numbers and the set of not practical numbers.

However, we wondered if this division of $\mathbb{N}$ is useful, because we don't know how many practical numbers there are. It is possible to prove that the first practical number are 1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48, 54... but the question is: how many are they?

We will prove they are infinite and in order to do this we follow the proof for the prime numbers offered by Euclide and published in his work Elements (Book IX, Proposition 20).

Theorem 3.1.2. *(Euclide's Theorem) Prime numbers are more than any assigned multitude of prime numbers.*

Proof. Let's assume that $p_1, p_2, ..., p_n$ is any finite list of prime numbers. We want to show that there is one additional prime number that is not in this list. Let P be the product of all the prime numbers in the list, that is $P = p_1 \cdot p_2 \cdot ... \cdot p_n$ and let $q = P + 1$.

- If q is a prime number, then there is at least one more prime that is not in the list.
- If q is not a prime number, then there is a prime factor p that divides q. So p divides q and also P then p would divide the difference of the two numbers, which is 1. Since no prime number divides 1, p cannot be on the list. This means that at least one more prime number exist beyond those in the list.

This proves that for every finite list of prime numbers there is a prime number not on the list. □

This Theorem states that the set of prime numbers is infinite.
Also the practical numbers are infinite, to prove this we have to introduce the following theorem:

Theorem 3.1.3. *The product of two practical numbers is a practical number.*

Proof. Given p_1 and p_2 practical numbers with $p_1 < p_2$,

- if $n \in \mathbb{N}$ and $n < p_1$ we can write n as a sum of divisors of p_1 but the divisors of p_1 are also divisors of $p_1 \cdot p_2$ therefore n is sum of divisors of $p_1 \cdot p_2$;
- if $n \in \mathbb{N}$ and $1 \leq n < p_2$ since p_2 practical, then n is sum of divisors of p_2 that is $n = \sum_{d_i \in D_{p_2}} d_i$ with D_{p_2} the set of divisors of p_2 then $p_1 n = \sum_{d_i \in D_{p_2}} p_1 d_i$ but all the $p_1 d_i$ are divisors of $p_1 p_2$ since the products between p_1 and a divisor of p_2 are divisors of $p_1 p_2$ (d_i divide p_2). Therefore all the $p_1 n$ can be written as a sum of divisors of $p_1 p_2$.
- if $n \in \mathbb{N}$ and n is between two $p_1 n$ consecutive $p_1 n_1$ and $p_1 n_2$ then it can be written as $n = p_i n_1 + \sum_{\delta_i \in D_{p_1}} \delta_i$. Since both the adds can be written as sun of divisors of $p_1 p_2$ then also n is a sum of divisors of $p_1 p_2$.

  Therefore each n less then $p_1 p_2$ is a sum of divisors of $p_1 p_2$ then $p_1 p_2$ is practical.

□

Basing on the Theorem 3.1.3 and following the proof of Euclide's Theorem we can state this theorem:

Theorem 3.1.4. *Practical numbers are more than any assigned multitude of practical numbers.*

Proof. Consider any finite list of practical numbers $p_1, p_2, ..., p_n$. It will be pointed out that there is at least one more practical number that is not on the list. Called P the product of all the practical number on the list $P = p_1 \cdot p_2 \cdot ... \cdot p_n$. P is practical for the Theorem 3.1.3, and P is different from all the practical numbers in the list.

Therefore there is one more practical number that is not in the list. This proves that for every finite list of practical numbers there is a practical number not on the list □

Hence the set of pratical numbers is infinite:

Definition 3.1.5. *A set A is said to be finite, if A is empty or there is $n \in N$ and there is a bijection $f : \{1, ..., n\} \to A$. Otherwise the set A is called infinite.*

Definition 3.1.6. *Two sets A and B are equinumerous or of the same cardinality if there exists a bijection $f : A \to B$. We then write $A \sim B$ or $|A| = |B|$.*

Definition 3.1.7. *A set A is called countably infinite if $|A| = |\mathbb{N}|$. A is countable if it is finite or countably infinite. If it is not countable it is uncountable.*

Theorem 3.1.8. *Every infinite subset of $\mathbb{N}$ is countably infinite.*

Theorem 3.1.9. *If A is a infinity subset of $\mathbb{N}$ then A is countable.*

Theorem 3.1.10. *The subset of $\mathbb{N}$ of practical numbers is infinite.*

Theorem 3.1.11. *The subset of $\mathbb{N}$ of practical numbers is countably infinite set i.e. with the same cardinality of $\mathbb{N}$ ($\aleph_0$).*

We can conclude the practical numbers are as many as the natural numbers and then a reasonable number for divide the set $\mathbb{N}$ and to use this method to expand a fraction.

The problem now is if the denominator is a not practical number.

3.2 Non-practical numbers

If the denominator b of the fraction is not a practical number we need, if it exists, a natural number k such that bk is a practical number.

We need these theorems:

Theorem 3.2.1. *The product of the first prime numbers is a practical number.*

Proof. We use the induction technique:

Base case is with $n = 2$ and is true because 2 is a practical number.

Inductive step: we show that if $p = 2 \cdot 3 \cdot ... \cdot p_n$ is practial then $p_T = 2 \cdot 3 \cdot ... \cdot p_n \cdot p_{n+1}$ is also practical.

- if $n \in \mathbb{N}$ and $n \leq p$ then is a sum of divisors of p for inductive hypothesis then is a sum of divisors of p_T (each divisors of p is divisors of p_T).
- if n is between two kp consecutive k_1p and k_2p then $n = \lambda + k_1p$ with $\lambda < p$ then λ is a sum of divisors of p_T and $k_1p = p_{n+1} \cdot q + r$ with $q < p$ and $r < p$ then also q and r are sums of divisors of p_T.

  p_{n+1} is prime then $p_{n+1}d_i$, with d_i divisor of p, is a divisor of p_T then $p_{n+1}q = p_{n+1}(d_1 \cdot ... \cdot d_m = p_{n+1}d_1 + ... + p_{n+1}d_m$ is a sum of divisors of p_T then kp is a sum of divisors of p_T then n is a sum of divisors of p_T.

  Therefore all the numbers less than p_T are sum of divisors of p_T then p_T is practical.

□

Theorem 3.2.2. *The product of a practical numbers by one of his divisor is a practical number.*

Proof. Given p a practical number and d its divisor,

- if $n \in \mathbb{N}$ and $n < p$ then n is a sum of divisors of p then n is a sum of divisors of pd (if a number divide p then divide pd),
- if n is between two kp consecutive k_1p and k_2p ($k_1, k_2 < d$) then $n = k_1p + r$ with $r < p$ then r is a sum of divisors of pd for the point above.

  d divide p then exists λ such as $k_1p = k_1\lambda d$ with $k_1\lambda < p$ then $k_1\lambda$ is a sum of divisors of p then $k_1\lambda = \sum_{d_l \in D_p} d_l$ then $k_1p = \sum_{d_l \in D_p} dd_l$. Since $d_l \in D_p \Rightarrow dd_l \in D_{pd}$ ($D_{pd} = D_p \cup d \cdot D_p$) then for all l $dd_l \in D_{pd}$ then k_1p is a sum of divisors of pd. Then n is a sum of divisors of pd.

Therefore all the numbers less than pd are sum of divisors of pd then pd is practical.

□

Theorem 3.2.3. *The product of powers of the first prime numbers is a practical number.*

Proof. This theorem is direct consequence of the theorems 3.2.1 and 3.2.2 indeed a product of powers of the first prime numbers is the product of the of the first prime numbers times some its divisors. □

The previous theorems guarantee that there is a factor that makes the denominator a practical number. We choose k equal to the product of the prime number in order to complete the list of the first prime of the decomposition of b.

Example:

$$\frac{a}{b} = \frac{7}{10}; \qquad b = 2 \cdot 5$$

Now, the first prime number that do not appear in the expansion of b is 3 and so $k = 3$. So, to have another equivalent fraction of the original one we have to multiply $\frac{a}{b}$ for $\frac{k}{k}$ and we get:

$$\frac{7}{10} = \frac{7}{10} \cdot \frac{3}{3} = \frac{21}{30}$$

The fact that we obtain 30 that is a practical number, let us express $\frac{a}{b}$ as sum of fractions with numerator a divisor of 30 over 30.

3.3 Observations

Our method is not perfect. It guarantees the possibility of finding one expansion of the fraction but we will not get all the existing expansions and it does not guarantee that it is the best, neither the shortest nor the one with the smallest denominators.

Example. Let the fraction $\frac{7}{8}$ where 8 is practical. With our method we can expand

$$\frac{7}{8} = \frac{4+2+1}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

but we can multiply the denominator by 3 and we get an other different expansion:

$$\frac{7}{8} = \frac{21}{24} = \frac{12+8+1}{24} = \frac{1}{2} + \frac{1}{3} + \frac{1}{24}$$

Example. Let the fraction $\frac{4}{7}$ where 7 is not practical. Our method states that we have to multiply the denominator by $2 \cdot 3 \cdot 5 = 30$. In this case we can multiply by 2. The denominator becomes 14 that is not a practical number but the new numerator is sum of divisors of 14 in fact $8 = 1 + 7$ so:

$$\frac{4}{7} = \frac{8}{14} = \frac{7+1}{14} = \frac{1}{2} + \frac{1}{14}$$

in order to find a practical denominator we have to multiply by 4 that is less than the number proposed by our method.

Example.

Let us follow others different ways:

- First of all we try our procedure.

$$\frac{3}{7} = \frac{3 \cdot 30}{7 \cdot 30} = \frac{90}{210}$$

$$\frac{90}{210} = \frac{1}{3} + \frac{1}{14} + \frac{1}{42}; \qquad \frac{90}{210} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35};$$

- Now we find another coefficient that makes the denominator practical.

$$\frac{3}{7} = \frac{3 \cdot 12}{7 \cdot 12} = \frac{36}{84}$$

$$\frac{36}{84} = \frac{1}{3} + \frac{1}{12} + \frac{1}{84}; \qquad \frac{36}{84} = \frac{1}{4} + \frac{1}{6} + \frac{1}{84};$$

- And we do it again using another different way.

$$\frac{3}{7} = \frac{3 \cdot 33}{7 \cdot 33} = \frac{99}{231}$$

$$\frac{99}{231} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$$

We notice that in every procedure the factor k is different. The first is $k = 30$ and it is obtained with our method, the second is $k = 12$ and it is lower than the first and the third is $k = 33$ that is higher than the first one. So we can state that the possible expansions are connected with the coefficient used.

So, we propose some different ways and we made a comparison with our method.

Theorem 3.3.1. *(A. Galletti e K.P.S. Bhaskara Rao [6]). The factorial of a natural number n is practical.*

Basing of this theorem we can say that exists at least one coefficient k for every initial denominator b, that multiplied for the given fraction let us get a practical denominator. Unfortunately this method to find k has big disadvantage because it is true that it proves that exist at least one k but does not care about the height of this number. For example, for our method we reach:

$$\frac{a}{b} = \frac{7}{10} \quad \text{and} \quad k = 3$$

Instead, using this theorem k would be $9!$ (which is 362.880), which is much greater than $b \cdot k$ $(7 \cdot 3 = 30)$.

For this reason we prefer to use our method based on theorems 3.2.1, 3.2.2 and 3.2.3 instead of the theorem of A.Galletti and K.P.S Bahaskara Rao. Notice that we can prove that theorem 3.3.1 comes from our method to find k. In fact, for example:

$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$$

$$9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7$$

So, a factorial number is the product of powers of the first prime numbers. For this reason it is a practical number for theorem 3.2.3.

An other result:

Theorem 3.3.2. *Let m be a practical number. If n is a integer such that $1 \leq n \leq \sigma(m) + 1$, then mn is a practical number, where $\sigma(m)$ denotes the sum of the positive divisors of m. In particular for $1 \leq n \leq 2m$, mn is practical.*

This result becomes from Stewart paper [1], in particular this theorem appears, for example, in [9] and it is the main tool to construct practical numbers. The first assertion follows from Stewart's structure theorem. Since $m - 1$ is a sum of distinct divisors of m, we have $m + (m - 1) \leq \sigma(m)$ then $2m \leq \sigma(m)$ and therefore, if $n \leq 2m$ then $n \leq \sigma(m) \leq \sigma(m) + 1$. (B.M. Stewart, 1954); The difference between this result to find k and our method is that the first one gave us the smallest number that multiplied for the initial denominator gets a practical number and the one that we use does not do this. For example, using the same example which we used before, for our method:

$$\frac{a}{b} = \frac{7}{10} \quad \text{and} \quad k = 3$$

and for B.M. Stewart method k would be the practical number greater than $\frac{b}{2}$:

$$k \geqslant \frac{b}{2} \Rightarrow k = 6$$

In conclusion none of this method give us all the expansion of a given irreducible proper fraction and so our method, even if does not give us the minor possible k, is fine. Also our strategy is fine because it has a great advantage: it always gives us at least a possible expansion.

3.4 The algorithm

The algorithm was written in C++ language, we wrote it because we needed to simplify and make faster every expansion of a given fraction into sum of Egyptian fractions. To make it happen the most important thing is to find all proper divisor of b, and then create a matrix which dimensions are n and $2^n - 1$ where n is the number of proper divisors of b. Here it is an example of this matrix:

		d_1	d_2	d_3	d_4
1	$\rightarrow$	0	0	0	1
2	$\rightarrow$	0	0	1	0
3	$\rightarrow$	0	0	1	1
4	$\rightarrow$	0	1	0	0
5	$\rightarrow$	0	1	0	1
6	$\rightarrow$	0	1	1	0
7	$\rightarrow$	0	1	1	1
8	$\rightarrow$	1	0	0	0
9	$\rightarrow$	1	0	0	1
10	$\rightarrow$	1	0	1	0
11	$\rightarrow$	1	0	1	1
12	$\rightarrow$	1	1	0	0
13	$\rightarrow$	1	1	0	1
14	$\rightarrow$	1	1	1	0
15	$\rightarrow$	1	1	1	1

Each column of the matrix is a proper divisor of b starting from left with the smaller one, and every row is a natural number starting from 1 until $2^n - 1$ but written in binary code. For example in the test matrix the last row corresponds to the natural number 15. Then with a loop, we check for each row if the sum of only divisors whose cell is equal to 1, is equal to the numerator. If it is true the program will print the expansion. Here there are many important pieces of our code

- the search of the proper divisors of b

```
for(int i=1; i<b; i++){
        if(b%i==0){
            div_b[n_div_b]=i;
            n_div_b++;
            cout<<i<<", ";
        }
    }
    cout<<endl;
```

- The creation of the matrix:

```
    int q,k;
    int lung=pow(2,n_div_b)-1;
    short casi[lung][n_div_b];
    for(int i=0; i<lung; i++){
        q=i+1;
        for(int j=n_div_b-1; j>=0; j--){
            k=j;
            if(q==1){
                casi[i][k]=1;
                while(k>0){
                    k--;
                    casi[i][k]=0;
                }
                break;
            }
            casi[i][j]=q%2;
            q=q/2;
        }
    }
```

- The expansion into Egyptian fraction

```
    for(int i=0; i<=lung; i++){
        for(int j=0; j<n_div_b; j++){
            if(casi[i][j]==1){
                somma+=div_b[j];    }
        }
        if(somma==a){
            cout<<a<<"/"<<b<<" = ";
            for(int h=0; h<n_div_b; h++){
                if(casi[i][h]==1){
                    cout<<"1"<<"/"<<b/div_b[h]<<" + ";
                }
            }
        cout<<endl; }
        somma=0;
    }
```

Chapter 4

Geometric method (our solution)

We now present a graphic method to expand a given irreducible proper fraction.

4.1 Egyptian fractions $\frac{1}{x} + \frac{1}{y}$

We choose to first deal with those fractions that can be written as the sum of two distinct unit fractions:

$$\frac{a}{b} = \frac{1}{x} + \frac{1}{y}$$

where a, b, x, y are non-zero natural numbers.

4.2 Fractions $\frac{sum}{product}$

Let's consider a fraction $\dfrac{s}{p}$ where s and p are such that $s = n + m$ and $p = x \cdot y$, with x and y natural numbers.

We observe that the sum $\dfrac{1}{x} + \dfrac{1}{y}$ is:

$$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{s}{p}$$

Example. Let $\dfrac{5}{6}$ be the fraction we want to write as a sum of two unitary fractions. We notice that

$$5 = 2 + 3 \quad \text{and} \quad 6 = 2 \cdot 3$$

hence we have the case where

$$\frac{s}{p} = \frac{5}{6} = \frac{1}{x} + \frac{1}{y} = \frac{1}{2} + \frac{1}{3}$$

.

4.3 Fractions $\frac{k \cdot sum}{k \cdot product}$

Let's consider the more general case of a fraction $\frac{a}{b}$ where a and b are relatively prime, such that

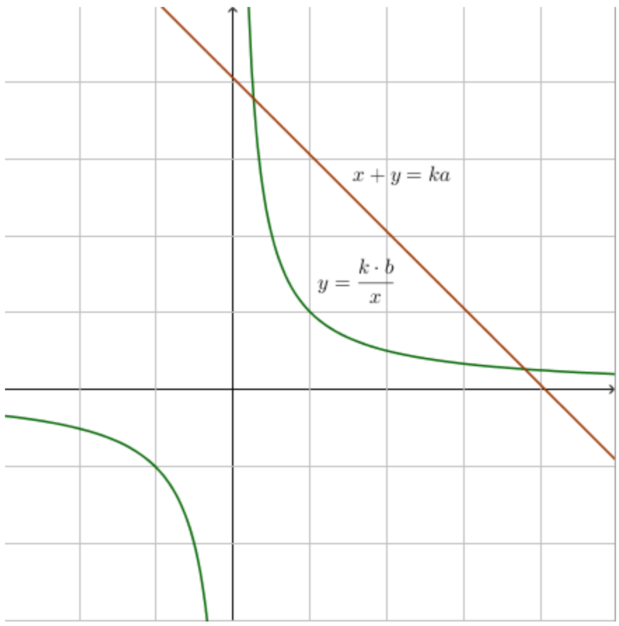
$$\frac{a}{b} = \frac{1}{x} + \frac{1}{y} = \frac{ka}{kb}$$

We want to find, if they exist as natural numbers, x and y satisfying the given condition, once we have chosen a and b.

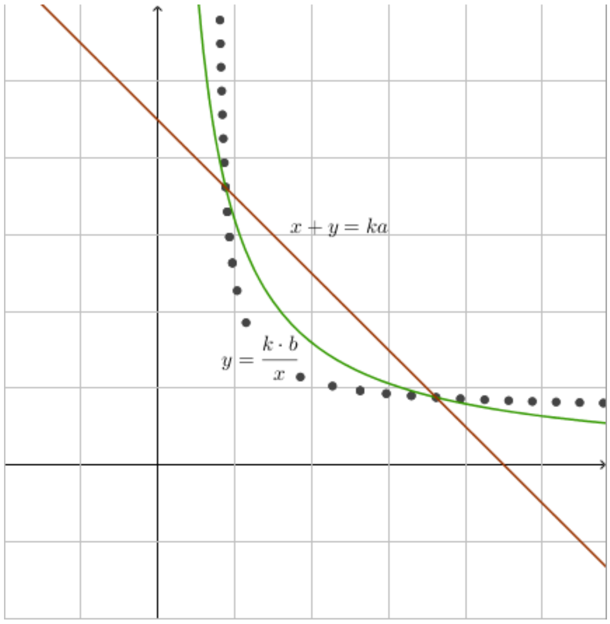
We can express the problem as a system of equations:

$$\begin{cases} x + y = ka \\ xy = kb \end{cases} \tag{4.1}$$

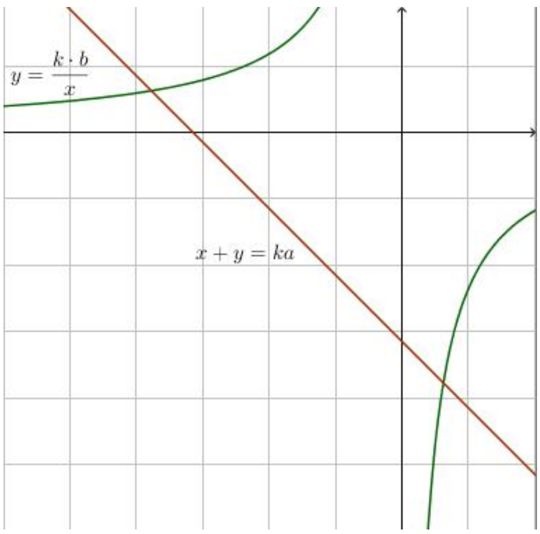
We decide to analyse the problem from a geometric point of view, so that the two equations correspond respectively to a sheaf of parallel straight lines and a sheaf of hyperbolas on the Cartesian plane $(x,\ y)$. The solutions are the intersection points between the two curves, with natural coordinates.



We notice that the solutions are positioned along a curve. If we change the value of k we can see that the curve moves.



At first it seems that in order to find the solution we have to consider all the values of k, that are infinite, but we notice that actually we must consider only $k > 0$. If k is negative we obtain a couple $(x;y)$ of discordant numbers because the intersection points belong to the second and fourth quadrant of the Cartesian plane, but we want x and y to be positive, because the Egyptians did not know negative numbers so we decide not to accept this case.



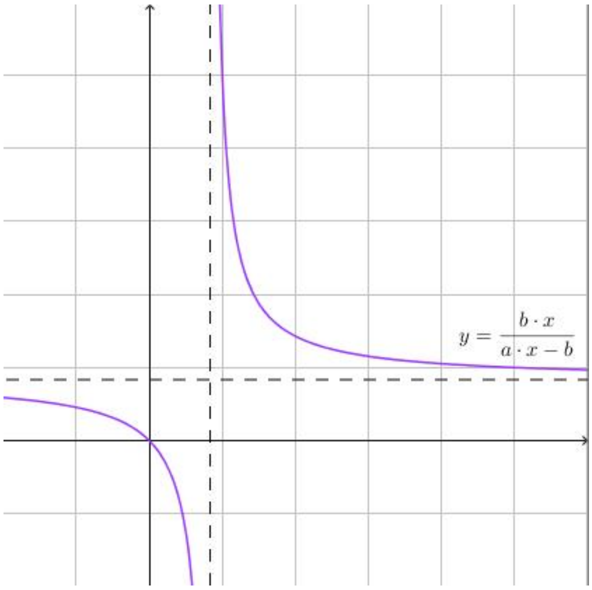
Another thing that can be noticed is that the abscissa of the solution is never less than a certain value. In fact, we find out that the geometric locus of the intersection points between the line $x + y = ka$ and the hyperbola $xy = kb$ is a homographic function:

$$y = \frac{bx}{ax - b}$$

and its asymptote is

$$x = \frac{b}{a}$$

which is exactly the value such that $x > \frac{b}{a}$.

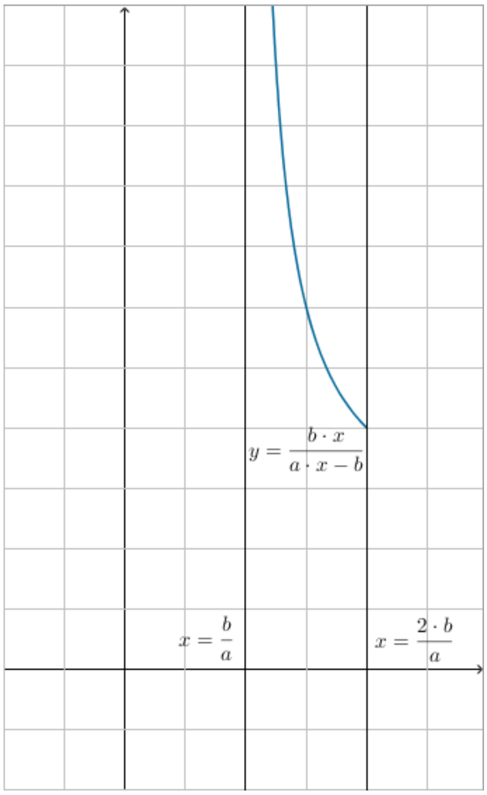


Furthermore, since the points of the geometric locus are symmetrical with respect to the axis $y = x$ we can consider just the range of the curve between the asymptote $x = \frac{b}{a}$ and the intersection point of the homographic function $y = \frac{bx}{ax - b}$ with $y = x$, that has abscissa:

$$x = \frac{2b}{a}$$

which means:

$$\frac{b}{a} < x < \frac{2b}{a}$$



We can get to the same result by following an algebraic method, starting form the same system of equations 4.1

$$\begin{cases} x + y = ka \\ xy = kb \end{cases}$$

which gives the following equation:

$$x^2 - kax + kb = 0 \tag{4.2}$$

From this equation we obtain the coordinates of the intersection points:

$$\left(\frac{ka - \sqrt{k^2a^2 - 4kb}}{2}; \frac{ka + \sqrt{k^2a^2 - 4kb}}{2} \right) \tag{4.3}$$

Hence the geometric locus of the intersection points between $x + y = ka$ and the hyperbola $xy = kb$ is:

$$y = \frac{1}{2} \left(\frac{ax}{ax - b} + \left| \frac{x(ax - 2b)}{ax - b} \right| \right)$$

But we must impose some conditions:

- $k^2a^2 - 4kb > 0$ (4.3) because of the square root ($\Delta \geq 0$) and because we want $x \neq y$. The result is $k > \frac{4b}{a^2}$ (we don't consider $k < 0$).

We already know that (4.2)

$$k = \frac{x^2}{ax - b}$$

so:

$$\frac{x^2}{ax - b} > \frac{4b}{a^2}$$

$$x > \frac{b}{a}$$

- $2x - ka < 0$ for $2x - ka = -\sqrt{k^2a^2 - 4kb}$ (4.3) must be coherent. The result is $k > \frac{2x}{a}$.

  We already know that (4.2)

$$k = \frac{x^2}{ax - b}$$

so:

$$\frac{x^2}{ax - b} > \frac{2x}{a}$$

$$\frac{ax^2 - 2bx}{ax - b} < 0$$

These conditions impose that:

$$\frac{b}{a} < x < \frac{2b}{a}$$

We notice that for these values the argument of the absolute value in the equation of the geometric locus is always negative, hence the final equation of the geometric locus is:

$$y = \frac{bx}{ax - b} \qquad \wedge \qquad \frac{b}{a} < x < \frac{2b}{a}$$

which is exactly the same relation that we found out using the geometric method.

After finding the values $x \in \mathbb{N}_0$ of the geometric locus that satisfy the condition $\frac{b}{a} < x < \frac{2b}{a}$ we need to check if the corresponding y is natural.

Theorem 4.3.1. *Let $\frac{a}{b}$ be an irreducible proper fraction, then it can be expanded as sum of two distinct unit fractions $\frac{1}{x} + \frac{1}{y}$, with $0 < x < y$, if*

$$\frac{b}{a} < x < \frac{2b}{a} \qquad \wedge \qquad y = \frac{bx}{ax - b}$$

Example: Let's consider $\frac{a}{b} = \frac{5}{18}$ that we want to write as $\frac{5}{18} = \frac{1}{x} + \frac{1}{y}$.
The integer numbers x such that $\frac{b}{a} < x < \frac{2b}{a}$ that is $\frac{18}{5} < x < \frac{36}{5}$ are:

- $x = 4 \rightarrow y = \dfrac{bx}{ax-b} = 36$, acceptable
- $x = 5 \rightarrow y \notin \mathbb{N}_0$
- $x = 6 \rightarrow y = \dfrac{bx}{ax-b} = 9$, acceptable
- $x = 7 \rightarrow y \notin \mathbb{N}_0$

So:

$$\frac{5}{18} = \frac{1}{4} + \frac{1}{36} = \frac{1}{6} + \frac{1}{9}$$

4.4 Egyptian fractions $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

We tried to retrace an analogous procedure for the expansion of a fraction in a sum of three unit fractions.

$$\frac{a}{b} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{ka}{kb}$$

Again, we can express the problem as a system of equations:

$$\begin{cases} xy + xz + yz = ka \\ xyz = kb \end{cases}$$

Each equation represents a surface on the Cartesian space (x, y, z) that changes for different values of k. From the system we determine the geometric locus of the solutions:

$$z = \frac{bxy}{axy - b(x+y)} \tag{4.4}$$

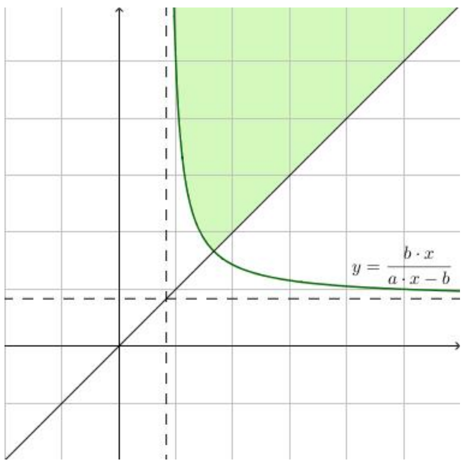
In this case we need some conditions on x, y, y to be able to solve the problem. We choose to impose $x < y < z$, expand the fractions $\frac{a}{b}$ starting from the greatest unitary fraction and following a descending order. This choice does not exclude any solutions, since imposing a different order of x, y, z would allow to find the same fractions in a different order, that is still acceptable thanks to the commutative property of the addition.

From the geometric locus (4.4) if we want $x, y, z > 0$ then:

$$axy - b(x+y) > 0$$

$$\Rightarrow \quad y > \frac{bx}{ax-b}$$

After, since the points of the first quadrant are symmetrical with respect to the $y = x$ axis, we are allowed to consider just the points that have $x < y$ and satisfy the previous condition.



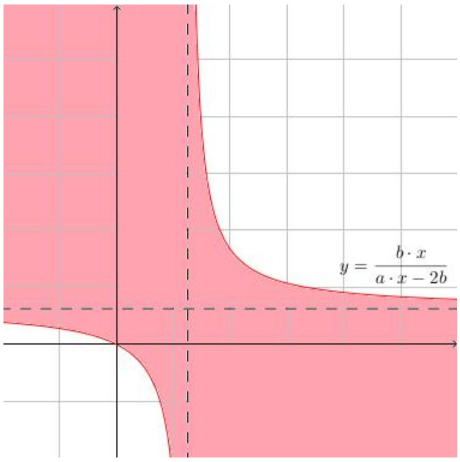
We can express z in 4.4 as a function of x and y and analise $x < z$:

$$\frac{bxy}{axy - b(x+y)} > x$$

$$ax^2y - bx^2 - 2bxy < 0$$

$$axy - bx - 2by < 0$$

$$\Rightarrow \quad y < \frac{bx}{ax - 2b}$$



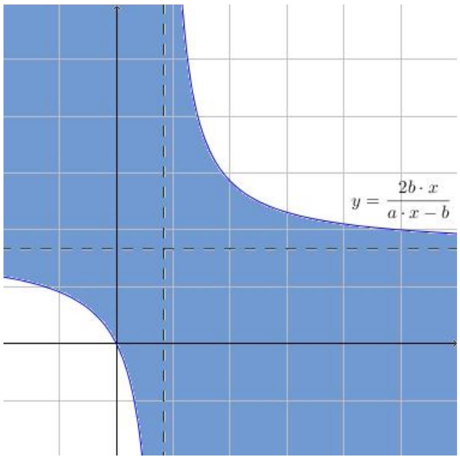
Then we proceed analising $y < z$:

$$\frac{bxy}{axy - b(x+y)} > y$$

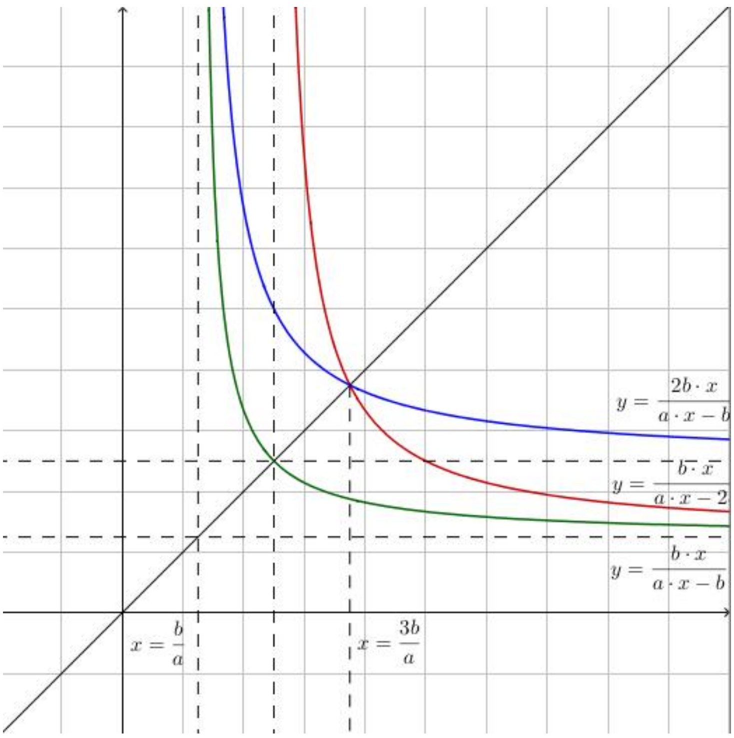
$$axy^2 - by^2 - 2bxy < 0$$

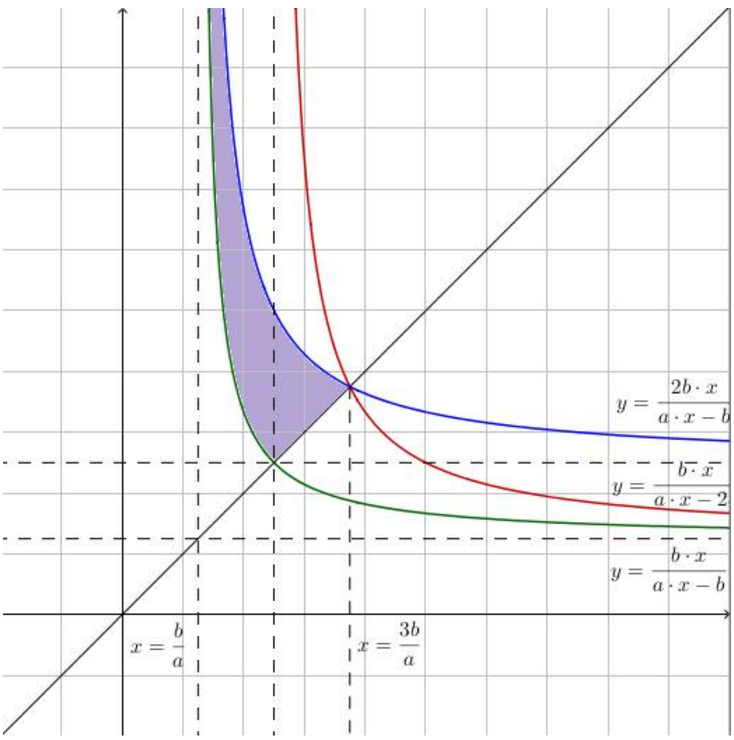
$$axy - by - 2bx < 0$$

$$\Rightarrow \quad y < \frac{2bx}{ax - b}$$



Representing on the same Cartesian plane all the graphs that we obtained, we can notice that the points (x,y) that satisfy all the conditions belong to a specific area, hence the corresponding coordinates z are positioned in a specific region of the space.





The area of the points (x,y) that satisfy all the conditions is not limited, but we must consider only the points with integer coordinates, therefore we are not considering an infinite number of points.

The graph shows that:

$$\frac{b}{a} < x < \frac{3b}{a} \quad \wedge \quad \frac{bx}{ax-b} < y < \frac{2bx}{ax-b}$$

which means that for every natural value of x we have to check if a natural value of y exists such that z is natural.

Theorem 4.4.1. *Let* $\frac{a}{b}$ *be a irreducible proper fraction, then it can be expanded as sum of three distinct unit fractions* $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ *with* $0 < x < y < z$, *if*

$$\frac{b}{a} < x < \frac{3b}{a} \quad \wedge \quad \frac{bx}{ax-b} < y < \frac{2bx}{ax-b} \quad \wedge \quad z = \frac{bxy}{axy - b(x+y)}$$

Example: Let's consider $\frac{a}{b} = \frac{4}{5}$ that we want to write as $\frac{4}{5} = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

The integer numbers x such that $\frac{b}{a} < x < \frac{3b}{a}$ that is $\frac{5}{4} < x < \frac{15}{4}$ are:

- $x = 2 \longrightarrow \frac{10}{3} < y < \frac{20}{3}$

  $\to\ y = 4 \to z = \frac{bxy}{axy-b(x+y)} = 20$, acceptable

  $\to\ y = 5 \to z = \frac{bxy}{axy-b(x+y)} = 10$, acceptable

  $\to\ y = 6 \to z \notin \mathbb{N}_0$

- $x = 3 \longrightarrow \frac{15}{7} < y < \frac{30}{7}$

  $\to\ y = 3 \to z \notin \mathbb{N}_0$ (not acceptable because$y = x$)

  $\to\ y = 4 \to z \notin \mathbb{N}_0$

So:

$$\frac{4}{5} = \frac{1}{2}+\frac{1}{4}+\frac{1}{20} = \frac{1}{2}+\frac{1}{5}+\frac{1}{10}$$

4.5 Egyptian fractions $\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}$: a general method

We tried to generalise the same method for the expansion of a fraction in a sum of n unit fractions:

$$\frac{a}{b} = \frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n} = \sum_{i=1}^{n}\frac{1}{x_i} = \frac{ka}{kb}$$

where the fractions are displaced in decreasing order $(0 < x_1 < x_2 < ... < x_n)$.

Again, we can express the problem as a system of equations that are represented by objects in the n-dimensions Cartesian space:

$$\begin{cases} \sum_{i=1}^{n}\frac{1}{x_i}\left(\prod_{j=1}^{n} x_j\right) = ka \\ \prod_{j=1}^{n} x_j = kb \end{cases} \tag{4.5}$$

Hence:

$$x_n = \frac{b}{a - b\sum_{i=1}^{n-1}\frac{1}{x_i}} \tag{4.6}$$

That we can see is coherently:

$$\frac{1}{x_n} = \frac{a - b\sum_{i=1}^{n-1}\frac{1}{x_i}}{b} = \frac{a}{b} - \sum_{i=1}^{n-1}\frac{1}{x_i} \tag{4.7}$$

We conclude that, similarly to what occurs for $n = 2$ and $n = 3$, in the most general case:

$$\frac{b}{a} < x_1 < \frac{nb}{a} \quad \wedge \quad \frac{b}{a - b(\frac{1}{x_1})} < x_2 < \frac{(n-1)b}{a - b(\frac{1}{x_1})} \quad \wedge \quad ...$$
$$\wedge \quad \frac{b}{a - b\left(\sum_{i=1}^{n-2}\frac{1}{x_i}\right)} < x_{n-1} < \frac{2b}{a - b\left(\sum_{i=1}^{n-2}\frac{1}{x_i}\right)}$$

therefore:

Theorem 4.5.1. *Let $\frac{a}{b}$ be a irreducible proper fraction, then it can be expanded as sum of n distinct unit fractions $\frac{1}{x_1} + ... + \frac{1}{x_n}$, with $x_0 < x_1 < ... < x_n$, $(x_0 = 0)$, if, for $j = 1, ...n-1$*

$$max\left\{x_{j-1}, \frac{b}{a - b\sum_{i=1}^{j-1}\frac{1}{x_i}}\right\} < x_j < \frac{(n+1-j)b}{a - b\sum_{i=1}^{j-1}\frac{1}{x_i}} \quad \wedge \quad x_n = \frac{b}{a - b\sum_{i=1}^{n-1}\frac{1}{x_i}}$$

Proof. Given n numbers $0 < x_1 < x_2 < ... < x_{n-1} < x_n$, we proved above that

$$x_n = \frac{b}{a - b\sum_{i=1}^{n-1}\frac{1}{x_i}}$$

Now, from $x_{n-1} < x_n$ then

$$x_{n-1} < \frac{b}{a - b\sum_{i=1}^{n-1}\frac{1}{x_i}} \quad \Rightarrow \quad ax_{n-1} - bx_{n-1} \cdot \sum_{i=1}^{n-2}\frac{1}{x_i} - b < b$$

$$\Rightarrow \quad \left(a - b\sum_{i=1}^{n-2}\frac{1}{x_i})\right)x_{n-1} < 2b \quad \Rightarrow \quad x_{n-1} < \frac{2b}{a - b\sum_{i=1}^{n-2}\frac{1}{x_i}}$$

Now, from $x_{n-2} < x_{n-1}$ then

$$x_{n-2} < x_{n-1} < \frac{2b}{a - b\sum_{i=1}^{n-2}\frac{1}{x_i}} \quad \Rightarrow \quad x_{n-2} < \frac{2b}{a - b\sum_{i=1}^{n-2}\frac{1}{x_i}}$$

and in the same way

$$x_{n-2} < \frac{3b}{a - b\sum_{i=1}^{n-3}\frac{1}{x_i}}$$

we continue in this way up x_1, in fact $x_1 < x_2 < \frac{(n-1)b}{a-b(\frac{1}{x_1})}$ and so

$$x_1 < \frac{(n-1)b}{a - b(\frac{1}{x_1})} \quad \Rightarrow \quad x_1 < \frac{nb}{a}$$

with this procedure it is possible to bound above all the variables.
Starting from $x_n > 0$ we get

$$\frac{b}{a - b\sum_{i=1}^{n-1}\frac{1}{x_i}} > 0 \qquad \Rightarrow \qquad \sum_{i=1}^{n-1}\frac{1}{x_i} < \frac{a}{b}$$

$$\Rightarrow \qquad \frac{1}{x_{n-1}} < \frac{a}{b} - \sum_{i=1}^{n-2}\frac{1}{x_i} \qquad \Rightarrow \qquad x_{n-1} > \frac{b}{a - b\sum_{i=1}^{n-2}\frac{1}{x_i}}$$

Now, $x_{n-1} > 0$ and

$$x_{n-1} < \frac{2b}{a - b\sum_{i=1}^{n-2}\frac{1}{x_i}} \qquad \Rightarrow \qquad \frac{2b}{a - b\sum_{i=1}^{n-2}\frac{1}{x_i}} > 0$$

$$\Rightarrow \qquad \sum_{i=1}^{n-2}\frac{1}{x_i} < \frac{a}{b} \qquad \Rightarrow \qquad x_{n-2} > \frac{b}{a - b\sum_{i=1}^{n-3}\frac{1}{x_i}}$$

continuing like this until $x_1 > \frac{b}{a}$. Then all the variables are bounded below.

Therefore all the inequalities of the theorem are valid for all $j = 1, ..n-1$ remembering that $\sum_{i=1}^{0}\frac{1}{x_i}$ is an empty sum, or nullary sum, that is a summation where the number of terms is zero. By convention the value of any empty sum of numbers is the the neutral element of addition that is zero.

We use the function $\max\left\{x_{j-1}, \frac{b}{a - b\sum_{i=1}^{j-1}\frac{1}{x_i}}\right\}$ as the left bound in order to eliminate unacceptable cases from the condition $x_1 < x_2 < ... < x_n$.

Conventionally we indicate $x_0 = 0$ to keep the correct formula correct for all indices j. $\square$

Example: Let's consider $\frac{a}{b} = \frac{3}{4}$ that we want to write as $\frac{3}{4} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$ which means $n = 3$. Let's proceed by:

- Calculating the range of possible x_1 values as:

$$\frac{4}{3} < x_1 < \frac{4 \cdot 4}{3}$$

- Determining the possible integer values of x_1 and calculating for each of them the possible ranges of x_2 values as:

$$\frac{4}{3 - 4\left(\frac{1}{x_1}\right)} < x_2 < \frac{3 \cdot 4}{3 - 4\left(\frac{1}{x_1}\right)}$$

- Determining the possible integer values of x_2 and calculating for each possible couple of x_1, x_2 the possible range of x_3 as (**??**):

$$\frac{4}{3 - 4\left(\frac{1}{x_1} + \frac{1}{x_2}\right)} < x_3 < \frac{2 \cdot 4}{3 - 4\left(\frac{1}{x_1} + \frac{1}{x_2}\right)}$$

- Determining the possible integer values of x_3 and calculating for each possible triplet of x_1, x_2, x_3 the values of x_4 by applying the formula 4.6 as:

$$x_4 = \frac{4}{3 - 4(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3})}$$

  or (4.7):

$$\frac{1}{x_4} = \frac{3}{4} - \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right)$$

- The integer values of x_4 mark the valid quadruplets of x_1, x_2, x_3, x_4 that satisfy

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = \frac{3}{4}$$

Following these steps we can find 27 solutions for the problem:

$$\tfrac{3}{4} =$$

$$= \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{21} + \tfrac{1}{420} \quad = \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{22} + \tfrac{1}{220} \quad = \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{24} + \tfrac{1}{120}$$

$$= \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{25} + \tfrac{1}{100} \quad = \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{28} + \tfrac{1}{70} \quad = \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{30} + \tfrac{1}{60}$$

$$= \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{36} + \tfrac{1}{45} \quad = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{13} + \tfrac{1}{156} \quad = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{14} + \tfrac{1}{84}$$

$$= \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{15} + \tfrac{1}{60} \quad = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{16} + \tfrac{1}{48} \quad = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{18} + \tfrac{1}{36}$$

$$= \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{20} + \tfrac{1}{30} \quad = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{21} + \tfrac{1}{28} \quad = \tfrac{1}{2} + \tfrac{1}{7} + \tfrac{1}{10} + \tfrac{1}{140}$$

$$= \tfrac{1}{2} + \tfrac{1}{7} + \tfrac{1}{12} + \tfrac{1}{42} \quad = \tfrac{1}{2} + \tfrac{1}{7} + \tfrac{1}{14} + \tfrac{1}{28} \quad = \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{72}$$

$$= \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{10} + \tfrac{1}{40} \quad = \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{12} + \tfrac{1}{24} \quad = \tfrac{1}{2} + \tfrac{1}{9} + \tfrac{1}{12} + \tfrac{1}{18}$$

$$= \tfrac{1}{2} + \tfrac{1}{10} + \tfrac{1}{12} + \tfrac{1}{15} \quad = \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{7} + \tfrac{1}{42} \quad = \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{24}$$

$$= \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{18} \quad = \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{10} + \tfrac{1}{15} \quad = \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{20}$$

4.6 Observation

This method finds the expansions only given a number of fractions of the sum but has the advantage that it finds all possible expansions in this case.

Chapter 5

Comparison

Now we want to compare the different methods that we have described in previous chapters. That is we will expand the same proper fraction with the four methods and will analyse the different results.

Let the proper fraction be $\frac{7}{9}$

1. **Fibonacci's method**

$$\frac{7}{9} = \frac{1}{2} + \frac{1}{4} + \frac{1}{36}$$

   - With this method we expand every proper fraction $\frac{a}{b}$ in **only one way**.
   - The numbers of unit fractions of the expansion is less than or equal to the numerator a.
   - The denominator can grow quite huge.

2. **Golomb's method**

$$\frac{7}{9} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36}$$

   - With this method we expand every proper fraction $\frac{a}{b}$ in **only one way**.
   - The numbers of unit fractions of the expansion is less than or equal to the numerator a.
   - The Golomb's algorithm is better than the Fibonacci-Sylvester algorithm in the sense that the in this case the denominators are guaranteed to be less than $b(b-1)$.

3. **Method of practical numbers**

$$\frac{7}{9} = \frac{1}{2} + \frac{1}{6} + \frac{1}{9}$$

   - With this method we expand every proper fraction $\frac{a}{b}$ in **at least one way**, as many ways as there are the different ways in which the numerator can be written as the sum of denominator divisors. The possible expansions are connected with the coefficient k used ($k = 2$ if the denominator is a practical number), if we choose more numbers we will have **more expansions**.
   - The numbers of unit fractions is less than or equal to the number of divisor of b or kb.
   - The denominators of the distinct unit fractions are less than or equal to the denominator b or kb.

We recall that if you have the prime factorization of the number n, then to calculate how many divisors it has, you take all the exponents in the factorization, add 1 to each, and then multiply these "exponents + 1"s together.

4. **Geometric method**

   - two unit fractions: impossible
   - tre unit fractions:

$$\frac{7}{9} = \frac{1}{2} + \frac{1}{4} + \frac{1}{36} \qquad \frac{7}{9} = \frac{1}{2} + \frac{1}{6} + \frac{1}{9}$$

- With this method we find **all the expansion** of every proper fraction $\frac{a}{b}$ fixed the number of unit fractions.
- The numbers of unit fractions must be fixed previously.
- We find all the expansions so the growing of the denominator is not determinable.

Chapter 6

The tree of fractions

We have already noticed in the previous chapters that the identity

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

holds for all $n \in \mathbb{N}_0$. This equality is known as the **"splitting relation"**.

Therefore, for example, $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ or $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$ and $\frac{1}{6} = \frac{1}{7} + \frac{1}{42}$.

We could go on and we will find a lot of fractions that we can arrange in a tree shape.

$$\frac{1}{2}$$

$$\frac{1}{3} \qquad \frac{1}{6}$$

$$\frac{1}{4} \qquad \frac{1}{12} \qquad \frac{1}{42} \qquad \frac{1}{7}$$

$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{13} \quad \frac{1}{156} \quad \frac{1}{1806} \quad \frac{1}{43} \quad \frac{1}{56} \quad \frac{1}{8}$$

... ...

Our goal is to find a formula to calculate the denominator of a fraction of the tree by knowing its position, that is the number of its row (r) and the number of its column (c).

The formula is: For all $r, c \in \mathbb{N}_0$ with $1 \leq c \leq 2^{r-1}$

$$A(r,c) = \begin{cases} 2 & \text{if } r = c = 1 \\ \left[A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right) + 1\right] \cdot \left[A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right)\right]^p & \text{if } r > 1 \end{cases}$$

with

$$p = 1 - \left\{c + \frac{1}{2} \cdot \operatorname{sgn}(r-2) \cdot \left[\operatorname{sgn}\left(c - 2^{r-2} - \frac{1}{2}\right) + 1\right]\right\} \bmod 2$$

Now we want to explain how we found it.

6.1 The formula

Looking for the formula above we met several problems:

- the recursion of the formula;
- the need to operate differently with the unit fractions $\dfrac{1}{n+1}$ and $\dfrac{1}{n(n+1)}$;
- the symmetry of the tree;
- the non-symmetry in the second row of the tree.

6.2 The recursion of the formula

Recursion means that in order to calculate one element of the tree we have to know the elements in the previous rows.

Each element of the tree is obtained from an element of the previous row $\Rightarrow r-1$. The two-by-two elements are obtained from the one in the previous row in the middle of them $\Rightarrow \left\lceil \frac{c}{2} \right\rceil$ that is we used the **ceiling function**.

It is necessary to start this process therefore we fixed at first $A(1,1) = 2$ and then $A(r,s)$ must be a function of $A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right)$ so

$$A\left(r,c\right) = \begin{cases} 2 & \text{if } r = c = 1 \\ f\left[A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right)\right] & \text{if } r > 1 \end{cases}$$

6.3 The need to operate differently with the unit fractions $\dfrac{1}{n+1}$ and $\dfrac{1}{n(n+1)}$

Now we have to find the function f. We notice that sometimes $f\left[A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right)\right]$ has to give $A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right) + 1$ and sometimes $\left[A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right) + 1\right] \cdot \left[A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right)\right]$

That is, in the second case we have to multiply by the factor $\left[A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right)\right]$ whereas in the first it is not necessary.

Therefore we put an exponent p in this factor which becomes 0 or 1.

To obtain this exponent we use a new function: **mod**, which gives the rest of the euclidean division between two integer numbers.

So the formula becomes:

$$A\left(r,c\right) = \left[A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right) + 1\right] \cdot \left[A\left(r-1, \left\lceil \frac{c}{2} \right\rceil\right)\right]^p$$

with $p = 1 - c \bmod 2$.

6.4 The symmetry of the tree

With the second formula we get a correct tree:

$$\frac{1}{2}$$

$$\frac{1}{3} \qquad\qquad\qquad\qquad \frac{1}{6}$$

$$\frac{1}{4} \qquad \frac{1}{12} \qquad\qquad \frac{1}{7} \qquad \frac{1}{42}$$

$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{13} \quad \frac{1}{156} \quad \frac{1}{43} \quad \frac{1}{1806} \quad \frac{1}{8} \quad \frac{1}{56}$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

But to have a more "elegant" tree we could build it in a symmetrical way. This symmetry is get by the location of the greatest fractions on the left for the first 2^{r-2} elements of the tree and on the right for the element from $2^{r-2}+1$ to 2^{r-1}.

Therefore we have to change the exponent p before and after 2^{r-2}. We want the follows exponents:

...
0 1 1 0
0 1 0 1 1 0 1 0
0 1 0 1 0 1 1 0 1 0 1 0
...

We are looking for a function that gives 0 or 1. We know the function

$$\frac{x}{|x|} = \begin{cases} -1 \text{ if } x < 0 \\ +1 \text{ if } x > 0 \end{cases}$$

and we can change it with a translation and a dilatation and we get

$$\frac{1}{2} \cdot \left(\frac{x}{|x|} + 1 \right) = \begin{cases} 0 \text{ if } x < 0 \\ +1 \text{ if } x > 0 \end{cases}$$

The change of the value of the exponent must happen after the column 2^{r-2} that is if $c - 2^{r-2} - \frac{1}{2} < 0$ then the exponent has to be 0 otherwise has to be 1. Then the function is

$$\frac{1}{2} \cdot \left(\frac{c - 2^{r-2} - \frac{1}{2}}{\left|c - 2^{r-2} - \frac{1}{2}\right|} + 1 \right) = \begin{cases} 0 \text{ if } c - 2^{r-2} - \dfrac{1}{2} < 0 \\ +1 \text{ if } c - 2^{r-2} - \dfrac{1}{2} > 0 \end{cases}$$

So the formula becomes:

$$A(r,c) = \left[A\left(r-1, \left[\frac{c}{2}\right]\right) + 1 \right] \cdot \left[A\left(r-1, \left[\frac{c}{2}\right]\right) \right]^{1 - \left[c + \frac{1}{2} \cdot \left(\frac{c - 2^{r-2} - \frac{1}{2}}{\left|c - 2^{r-2} - \frac{1}{2}\right|} + 1 \right) \right] \bmod 2}$$

6.5 The non-symmetry in the second row of the tree

The last problem is that the exponents in the second row are 0 and 1 hence they are not symmetric.

0 1
0 1 1 0
0 1 0 1 1 0 1 0
... ...

Then the last work is to correct the formula to obtain the non-symmetry only in the second row but to keep symmetry in the remaining rows.

Therefore we need a function that changes the exponents only from the second line onwards.

We use the **sign function**:

$$sgn\left(x\right)=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ +1 & \text{if } x>0\end{cases}$$

Then we introduce a factor that is 0 or 1 to produce a change of the order of the fractions only from the second rows onwards. This factor is $(r-2)$. So

$$sgn\left(r-2\right)=\begin{cases}-1 & \text{if } r<2 \quad \text{(it never happens)}\\ 0 & \text{if } r=2\\ +1 & \text{if } r>2\end{cases}$$

The case $r<2$ it never happens in our tree. If $r=2$ the symmetry is cancelled, in the other case the symmetry remain because the result of this function is equal to 1.

The formula now becomes:

$$A\left(r,c\right)=\left[A\left(r-1,\left\lceil\frac{c}{2}\right\rceil\right)+1\right]\cdot\left[A\left(r-1,\left\lceil\frac{c}{2}\right\rceil\right)\right]^{1-\left[c+\frac{1}{2}\cdot\text{sgn}(r-2)\cdot\left(\frac{c-2^{r-2}-\frac{1}{2}}{\left|c-2^{r-2}-\frac{1}{2}\right|}+1\right)\right]\text{mod } 2}$$

At last we noticed that, for $x\neq 0$ $\text{sgn}(x)=\dfrac{x}{|x|}$ and then we can transform the exponent into the final form:

$$1-\left\{c+\frac{1}{2}\cdot\text{sgn}(r-2)\cdot\left[\text{sgn}\left(c-2^{r-2}-\frac{1}{2}\right)+1\right]\right\}\text{ mod } 2$$

6.6 The expansion of the natural number

With the result of the previous section we can expand the number 1 into a selected number of unit fraction.

Indeed $1=\dfrac{1}{2}+\dfrac{1}{2}$ so we can expand the second addend knowing that the sum of all the fraction in the second row of the tree is $\dfrac{1}{2}$. In this way we expand 1 into three unit fraction.

$$1=\frac{1}{2}+\frac{1}{2}=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$$

Also the sum of the fraction in the third row is equal to $\dfrac{1}{2}$ so we can use this row too but this time what we obtain is an expansion with five unit fraction.

$$1=\frac{1}{2}+\frac{1}{2}=\frac{1}{2}+\frac{1}{4}+\frac{1}{12}+\frac{1}{42}+\frac{1}{7}$$

The sum of the fraction in a random row is always equal to $\dfrac{1}{2}$ so we can expand 1 in infinite way but the number of unit fraction change.

$$
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & \frac{1}{2} & & & & & & & & & +\frac{1}{2} = & 1\\
 & & & \frac{1}{3} & & & & & + & & & \frac{1}{6} & & & & & & +\frac{1}{2} = & 1\\
 & \frac{1}{4} & & + & & \frac{1}{12} & & & + & & \frac{1}{42} & & + & & \frac{1}{7} & & & +\frac{1}{2} = & 1\\
\frac{1}{5} + \frac{1}{20} & + & \frac{1}{13} & + & \frac{1}{156} & & & & + & & \frac{1}{1806} & + & \frac{1}{43} & + & \frac{1}{56} + \frac{1}{8} & & & +\frac{1}{2} = & 1\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & & & & & & & & &
\end{array}
$$

$\frac{1}{2}$ can be obtained also summing different part of different row of the tree.
For example:

$$1 = \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{6}\right) = \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{12}\right) + \frac{1}{6} = \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{20}\right) + \frac{1}{12} + \frac{1}{6}$$

If we sum two rows the result is 1 because the sum of each row is equal to $\frac{1}{2}$.
For instance if we sum the first and the second row:

$$\left(\frac{1}{3} + \frac{1}{6}\right) + \left(\frac{1}{4} + \frac{1}{12} + \frac{1}{42} + \frac{1}{7}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

In the same way if we sum a pair number of rows we will get a natural number.

Let n be the natural number that we want to get then the number of rows r that we have to sum is $2n$.

The choosing of the rows is irrelevant: the result is always the same if I keep a fixed number of rows.

This brings the consequence that also all the number in $\mathbb{Q}_0^+$ can be expanded.

Indeed an improper fraction is always the sum a natural number plus a proper fraction, this last can be expanded with the method we prefer. The only restriction is the choice of rows without unit fractions used for the expansion of the proper fraction.

Answer:
All the natural numbers can be expanded into distinct unit fractions.
Also the improper fractions can be expanded into distinct unit fractions.

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