Liceo scientifico Ippolito Nievo, Padua

Project Math en. Jeans

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Is there any strategy that allows Heracles to kill the hydra?

Definitions

- Maximum node (a): every head is a *maximum node*;
- Value of a node: a node at height h has value v if it is connected to v *maximum nodes* at height $h+1$;
- Maximum node (b): a non-head node at height h is a *maximum node* if it has the highest value among all of the nodes at height h.
- Vector: we associate to every hydra of height n a vector with n components. The i-th component of this vector is the maximum among the values of the nodes at height n-i.
- Order relation between vectors: a vector is smaller than another vector if it has less components or the first different component they have is smaller. Formally, $(a_1, \ldots, a_n) < (b_1, \ldots, b_m)$ if $n < m$, or $n = m$ and there exists an index i such that $a_i < b_i$ and $a_j = b_j$ for every $j < i$.

Procedure

Beginning from the root of the hydra, following a path of maximum nodes, up the maximum height, we cut the head we reach. We will eventually defeat the hydra by iterating this procedure finitely many times.
We prove that, at each move, the vector associated to any hydra decreases, and that every decreasing sequence of vectors is finite. In particular, the effect of the procedure is that the first component from the right greater than 1 decreases by 1, or, if every component is equal to 1, the vector gets shorter.

If the vector has every component equal to 1, we have online one head at the maximum height (otherwise, we could consider the paths of maximum nodes ending at two such heads, and the node at which the two paths connect would have value greater than 1). Therefore, cutting the only head at the maximum height, the vector will shorten up by 1.

Let's evaluate how the vector changes after the cut, if it has some components different from 1. It's important to observe that if the vector has $k < n$ $(k \geq 0)$ components from the right equal to 1, then, at heights greater than k, the maximum node is not unique. When we cut the head at the end of a maximum path, the value of the node right beneath it will decrease by 1 and, having other maxima at that height, this node is no longer a maximum.

Likewise, the value of the node right beneath this node will decrease by 1 and it won't be a maximum anymore. This pattern repeats itself up to the maximum node at height *k*, which is the only one at this height. So, it turns out to be still a maximum node (possibly not unique).
We can now observe that, on maximum paths different from the one considered so far, the values of the nodes at a height between *n-1* and *k+1* are unchanged and the node at height *k* decreased by 1.

Therefore we just need to show that the replication does not change these values. If the replication stimulus comes from a node which is not a maximum, the node's value right beneath it does not change and does not become a maximum. Hence neither do all the nodes underneath. Also, in every replication, the maximum value at heights that are greater than the replicating node remain unchanged, since the nodes that are generated are copies of pre-existing nodes at that height. Let's notice then that the replication in this case leaves the first *n-k* components of the vector unvaried, since up to *k* the replication stimulus proceeds in non-maximum nodes (that can't become such during the replication).

Lastly, let's prove by induction that no infinite decreasing sequence of vectors exists, so that a hydra cannot survive a large enough (finite) number of attacks.

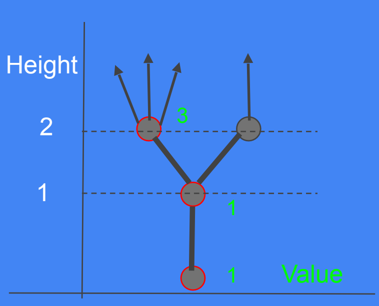
Write P(n) for "An infinite decreasing sequence of vectors with *n* components does not exist" .
Basis: n=1. An infinite decreasing sequence of positive integers does not exist (well-ordering principle).
Inductive hypothesis: We assume that P(n) holds for an arbitrary natural number *a* (P(*a*)).
Inductive step: We prove that P(a+1) holds .
For fixed first *a* components of a vector with length *a+1*, the last *(a+1)-th* component can only decrease for a finite amount of times. Therefore, given any infinite decreasing sequence *S* starting with a vector with *a+1* components, the sequence *S'* obtained by considering only the first *a* components of vectors in *S* would contain an infinite decreasing sequence of vectors of length *a.* But this contradicts the inductive hypothesis.
Q.E.D.

In the following slides,"knot" and "node" are synonyms.

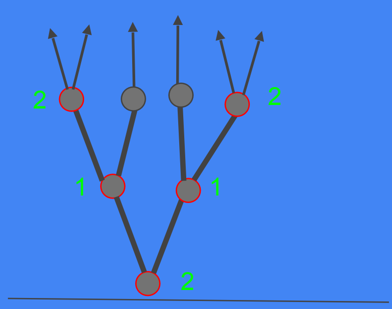
How to kill the Hydra

Value - Maximum Knot

Value: the value of a knot at height *h* linked to *n* maximum knots at height *h+1* it's equal to *n*.
Maximum Knot: a knot is a maximum knot if it has the maximum value among all the knots at the same height (every head at the maximum height is a maximum knot).



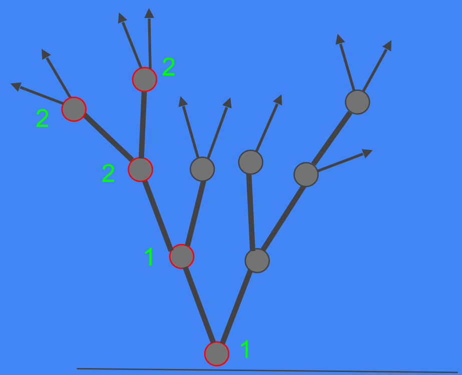
(3,1,1)



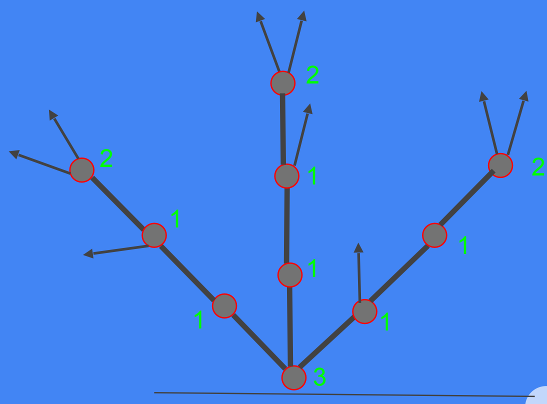
(2,1,2)

How to identify an Hydra

Every Hydra corresponds to a vector (a,b,c,...,z). Every number of the vector corresponds to the value of the maximum knot(s) going from the top to the bottom.



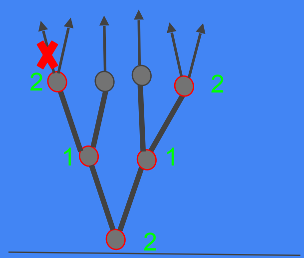
(2,2,1,1)



(2,1,1,3)

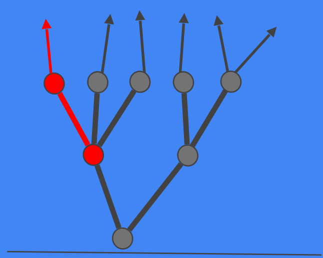
What happens when you cut off a head

Before cutting the head

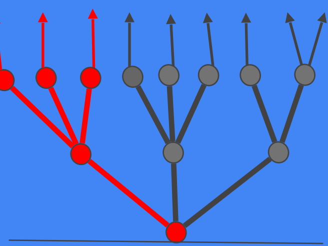


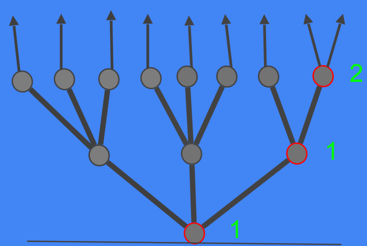
(2,1,2)

Step 1 of replication



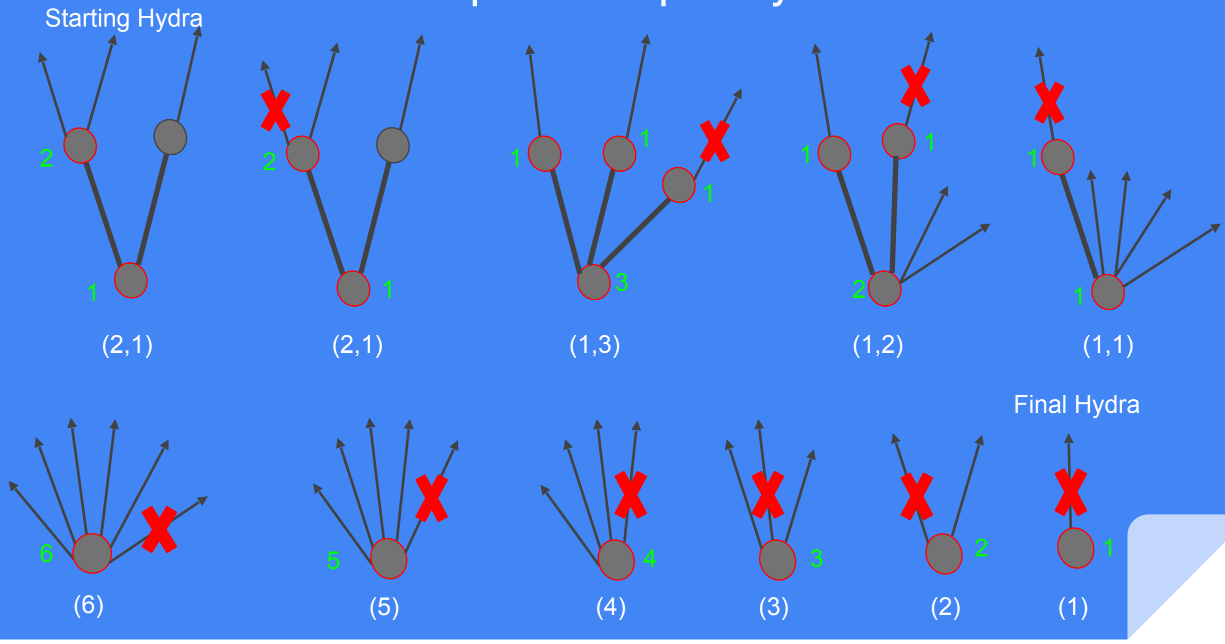
Step 2 of replication





(2,1,1)

Example - Simple Hydra



Induction

P(n): It does NOT exist an infinite decreasing series of vectors of *n* components.

P(1): It does NOT exist an infinite decreasing series of natural numbers.

We assume P(n) as the *hypothesis* and P(n+1) as the *thesis*.

Given that the first *n* components will not be altered, the last component can decrease only a finite number of times.

(*Demonstration by absurd*)

In an infinite series the vector composed of the first *n* components must decrease an infinite number of times. And this is absurd by the *hypothesis* P(n).