Coloring the Plane

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1 Coloring the Real Plane

The problem dates to 1950, when Edward Nelson, then student at the University of Chicago, posed to John Isbell, a fellow student, the following question: *How many colors are needed to color all the points in the Euclidean plane so that no two points at distance one apart have the same color?*
Nelson himself proved that 4 colors are needed, Isbell proved that 7 colors are sufficient ([1]), but neither estimate has been improved since.

1.1 The three color plane problem

We discuss the following problem concerning the existence of coloring of the plane with 3 colors that satisfy *unit distance* condition:

Problem 1. Can the points of the Euclidean plane be colored with three colors, blue, red, green in such a way that no two points at distance one apart have the same color?

We'll assume that exists a coloring of the real plane that satisfies the hypothesis:

(**H**) All the points of the plane are colored with three colors, blue, red, green such that no two points of the same color are 1 distance apart.

We'll explore the consequences of the hypothesis (H) using two notions:

Definition. A set of points of the plane is

- *monochromatic* if all the points of the set are of the same color
- *2-chromatic* if all the points of the set are colored with 2 colors exactly.

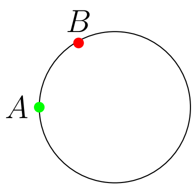
We examine simple sets of points of the plane, such us lines and circles. Could be exists, under the conditions of hypothesis (H), some monochromatic line or some monochromatic circle?

Fact 1. Let us assume hypothesis (H). Then every line of the plane is not monochromatic.

In fact, if A, B are two points of a line r that are 1 distance apart, then A, B must have different colors.

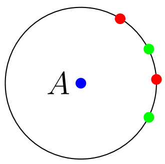
Fact 2. Let us assume hypothesis (H). If a circle has radius r such that $r > \frac{1}{2}$ than it is not monochromatic.

Let us assume that the length of the diameter of circle $\mathcal{C}$ is greater than 1, so that we can find two points A, B on $\mathcal{C}$ such that they are 1 distance apart. Then colors of A, B must be different.



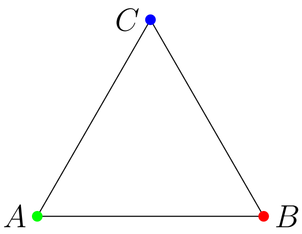
The hypothesis about radius r of $\mathcal{C}$ is equivalent to say $2\pi r > 1$, i. e. $r > \frac{1}{2\pi}$.

Fact 3. Let us assume hypothesis (H). If a circle of the plane has radius 1 then it is 2-chromatic.



In fact, let us assume that A is the center of a circle $\mathcal{C}$ with radius 1. If color of A is blue, then the color of every point of $\mathcal{C}$ must be *green* or *red*. Fact 2 assures us that 2 colors are necessary so we can conclude that $\mathcal{C}$ is 2-chromatic.

Fact 4. Let us assume hypothesis (H). Then the vertices of every equilateral triangle of side 1 have three distinct colors.

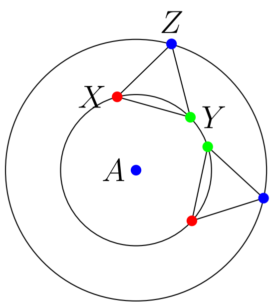


Fact 5. Let us assume hypothesis (H). Then there exists a monochromatic circle with radius > 1.

In fact, let us assume that A is the center of a circle $\mathcal{C}$ with radius 1. Fact 3 assures it is 2-chromatic circle, let its points be green or red. For every couple of points (X, Y) of circle $\mathcal{C}$ such that $XY = 1$ let be $Z = f(X, Y)$ the point exterior to circle $\mathcal{C}$ such that triangle XYZ is equilateral. Than the set of points

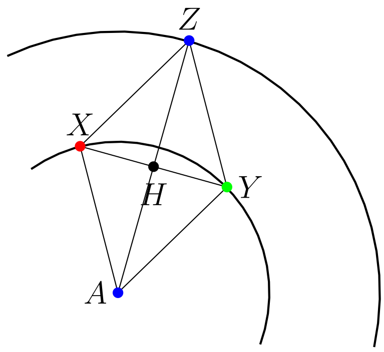
$$\{f(X, Y) : X, Y \in \mathcal{C}\}$$

is a circle with radius $\sqrt{3}$.



In fact,

- if X, Y are points of $\mathcal{C}$, at distance 1 apart, and H is the midpoint of X, Y, then the three points $A, H, Z = f(X, Y)$ are aligned on the axis of X, Y, the two triangles AXY, XYZ are symmetric around line XY and the length of AZ is the double of $\frac{\sqrt{3}}{2}$, the length of the height HZ of unitary equilateral triangle XYZ.

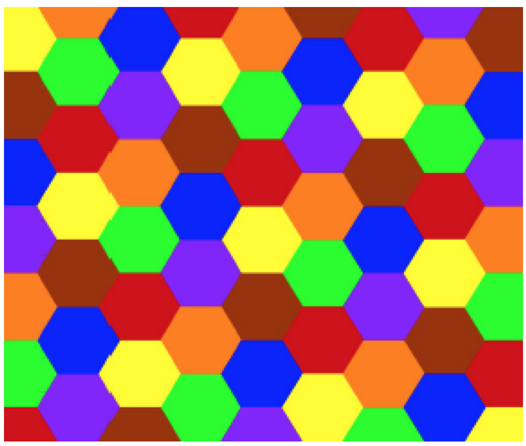


- Let be Z a point on the circle $\mathcal{C}$ of center A and radius $\sqrt{3}$. Let us consider the two points X, Y on circle $\mathcal{C}$ at distance 1 from Z. Then Z, A are on the axis of X, Y so that triangle XYZ is the reflection of triangle AXY around line XY. As a consequence, XYZ is a triangle equilateral, with side equal to $AX = 1$, so that $Z = f(X, Y)$.

Now, we see that Fact (5) contradicts Fact (2) and they are two consequnces of hypothesis (H) that we must refute. Problem 1 has no solution, so chromatic number of real plane must be greater than 3.

1.2 The seven color plane problem

An upper estimate of $\chi(\mathbb{R}^2)$, the minimum number of colors necessary to color all the points of the Euclidean Plane, was found by John Isbell with a tessellation of the Euclidean plane with regular hexagons:



We color one hexagon with yellow color and the six neighbors with six more colors. The figure composed of these 7 hexagons tessellates the plane, and it can easily be seen that distance 1 is never realized between points of the same color, providing the hexagons are of radius slightly less than $\frac{1}{2}$.

2 Coloring subsets of real plane

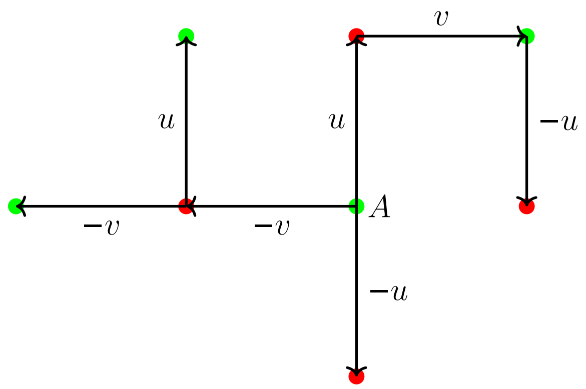
Let's suppose that S is a set of points in the real plane. The *chromatic number* of S, denoted $\chi(S)$, is defined as the minimum number of colors necessary to color all the points of S so that no two points in S that are at distance 1 have the same color.

2.1 Coloring the Integer Plane

Two points $A, B \in \mathbb{Z}^2$ are at distance 1 if and only if there exists a unitary vector w, with integer coordinates, such that $A + w = B$. So we'll consider the following relation $\sim$ defined for all points $A, B \in \mathbb{Z}^2$:

- $A \sim B$ if there exists a finite sequence $v_1, v_2, \ldots, v_n$, of unit vectors (i. e. with length 1), with integer coordinates, such that $B = A + v_1 + v_2 + \cdots + v_n$.

There are only 4 unit vectors in the Integer Plane $\mathbb{Z}^2$, i. e. $u = (1, 0), v = (0, 1), -u = (-1, 0), -v = (0, -1)$.



We define a notion of parity on the set of the integer vectors:

- an integer vector is said *even* if its coordinates have the same parity
- an integer vector is said *odd* if its coordinates don't have the same parity

For instance, (1, 2) and (-2, 1) are odd vectors, while (3, 5) and (4, -2) are even vectors. The definition could be reformulated as

- an integer vector v is *even* if and only if it is a sum $v = v_1 + v_2 + \cdots + v_n$ of an even number of unit integer vectors,
- an integer vector v is *odd* if and only if it is a sum $v = v_1 + v_2 + \cdots + v_n$ of an odd number of unit integer vectors,

We can color the set of point Z^2 using the following procedure. Let $A \in \mathbb{Z}^2$ an arbitrary point of the Integer Plane, then for every integer vector v, we assign the color of the point $A + v$ in such a way that

- if v is even then the color of $A + v$ is *green*
- if v is odd then the color of $A + v$ is *red*

2.2 Coloring the Rational plane

We want to proof the following result ([2]):

Theorem (Woodall, 1973). $\chi(\mathbb{Q}^2) = 2$.

In other terms, the rational points of the plane can be colored with only two colors in such a way that no two points that are unit distance apart have the same color. We'll base our proof using the same idea of parity used in the coloration of the Integer Plane. We begin observing that two points $A, B \in \mathbb{Q}^2$ are at distance 1 if and only if there exists a unit rational vector v such that $A + v = B$. So we'll consider the following relation $\sim$ defined for all points $A, B \in \mathbb{Q}^2$:

- $A \sim B$ if there exists a finite sequence $v_1, v_2, \ldots, v_n$, of unit rational vectors such that $B = A + v_1 + v_2 + \cdots + v_n$.

Let us adopt the symbol $\mathbb{U}_{\mathbb{Q}}$ to denote the set of all finite sums of unit rational vectors. The set $\mathbb{U}_{\mathbb{Q}}$ is much more intricate than the set of the sums of the unit integer vectors that coincide with the set of all integer vectors.

We note that

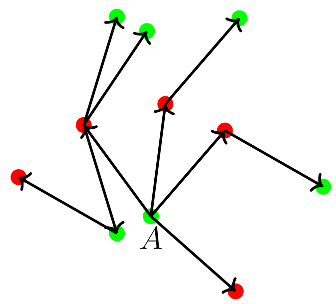
- $\sim$ is an equivalence relation on the rational plane $\mathbb{Q}^2$.
- if two rational points A, B are at distance 1 then $A \sim B$

Based on these observations let us restrict the problem of coloring the rational plane $\mathbb{Q}^2$ to coloring the points of each equivalence class. So we have reduced us to

Problem 2. Proof that if $\mathbb{E}_A$ is the equivalence class of an arbitrary rational point A, then $\chi(\mathbb{E}_A) = 2$.

To solve problem 2, we could proceed informally as follows:

- $A \mapsto$ *green*
- $A + u \mapsto$ *red*, for every unit rational vector u
- $A + u + v \mapsto$ *green*, for every unit rational vectors u, v
- ...



We used *parity* as a tool for classify integer vectors that are sum of unit vectors so we would like to have a procedure such as:

- $A + v \mapsto$ *red*, for every vector $v \in \mathbb{U}_{\mathbb{Q}}$ that is a sum of an odd number of unit rational vectors
- $A + v \mapsto$ *green*, for every vector $v \in \mathbb{U}_{\mathbb{Q}}$ that is a sum of an even number of unit rational vectors
- ...

But we are not sure that this procedure is well founded. In fact we don't know if there exists some rational vector that is both a sum of an odd number of unit vectors and a sum of an even number of unit vectors. We'll proceed in a series of steps to proof that the set $\mathbb{U}_{\mathbb{Q}}$ can be divided in even vectors and odd vectors, and that this is an effective partition of the set.

We'll reduce the notion of parity of a vector to a notion of parity of its coordinates. First step is finding a useful notion of parity of a unit rational vector. It will be based on a notion of parity of a rational number.

Definition. A rational number $q = \frac{a}{b}$ is said a *reduced form* if integers a, b are coprime and $b > 0$. We'll say that a is the *numerator* and b is the *denominator* of q.

Lemma 1. The coordinates of a unit rational vector are rational numbers with odd denominator while their numerators have different parity.

Proof. Let the suppose that the coordinates of a unit rational vector (x, y) are written in reduced form: $x = \frac{a}{b}, y = \frac{c}{d}$). Then $(\frac{a}{b})^2 + (\frac{c}{d})^2 = 1$, so $a^2d^2 + b^2c^2 = b^2d^2$, and b^2 divides a^2d^2, hence b divides d, as a, b are coprime. Similarly, d divides b, so $b = d$, and we have $a^2 + c^2 = b^2$.
Let us suppose that b is even. Then a, c are both odd, so reasoning modulo 4 we find that a^2 mod 4 + c^2 mod 4= b^2 mod 4, i. e. 1+1= 0 mod 4, a contradiction. So b cannot be even. Because b is odd and $a^2 + c^2 = b^2$ we have that a, c have different parity.

We'll concentrate our study to the set $\mathbb{D}$ of rational numbers with odd denominator. All the integer numbers n are in $\mathbb{D}$, as $\frac{n}{1}$ is the reduce form of n. Now, we'll extend the usual notion of parity from the set of Integers to the set $\mathbb{D}$:

Definition. If $q = \frac{a}{b} \in \mathbb{D}$ is a rational number written in reduced form, with odd denominator. We'll say that

- q is *even* if its numerator a is even
- q is *odd* if its numerator a is odd

This notion of parity satisfies the usual properties of parity over integer numbers:

Lemma 2. Let q_1, q_2 two rational numbers in set $\mathbb{D}$. Then $q_1 + q_2$ satisfies the parity rules:

- If q_1 and q_2 have the same parity then $q_1 + q_2$ is even.
- if q_1 and q_2 don't have the same parity then $q_1 + q_2$ is odd.

Proof. The same pattern of reasoning is common to all four cases, so we describe here only the case when q_1 is odd and q_2 is even. Let be $q_1 = \frac{a_1}{b_1}, q_2 = \frac{a_2}{b_2}$, so that a_1, b_1, b_2 are odd integers and a_2 is an even integer. Then $q_1 + q_2 = \frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1 b_2 + b_1 a_2}{b_1 b_2}$ is an odd rational number because it has an odd denominator $b_1 b_2$ and an odd numerator $a_1 b_2 + b_1 a_2$.

Lemma 3. If vector $w = (x_w, y_w)$ is the sum of two unit vectors then x_w, y_w are both elements of $\mathbb{D}$ and have the same parity.

Proof. Let us suppone that $w = u + v$ is the sum of two unit vectors u, v.

- $u = (x_u, y_u), v = (x_v, y_v)$ and the reduced forms of x_u, y_u, x_v, y_v have odd denominator, hence $x_w = x_u + x_v, y_w = y_u + y_v$ have an odd denominator, so $x_w, y_w \in \mathbb{D}$.
- Parity of x_u is different from parity of y_u and parity of x_v is different from parity of y_v. Let us suppose that $u = (even, odd), v = (odd, even)$. Then, for lemma 2, we have $u + v = (even + odd, odd + even) = (odd, odd)$. The three remaining cases use the same reasoning.

Now we define a notion of parity for a vector $v \in \mathbb{U}_{\mathbb{Q}}$:

- if coordinates of v have the same parity we say that v is *even*
- if coordinates of v don't have the same parity we say that v is *odd*

Lemma 4. Let $v, w \in \mathbb{U}_{\mathbb{Q}}$. Then $v + w$ satisfies the parity rules:

- If v and w have the same parity then $v + w$ is even.
- if v and w don't have the same parity then $v + w$ is odd.

A consequence is that the even vectors and odd vectors form a partition of the set $\mathbb{U}_{\mathbb{Q}}$:

Lemma 5. Let $v \in \mathbb{U}_{\mathbb{Q}}$. Then

- If v is a sum of an even number of unit vectors then it is even.
- if v is a sum of an odd number of unit vectors then it is odd.

We can color any point $P = A + v$ of the equivalence class E_A of a point A using the following procedure:

- If vector v is even then color $(P) = green$

- if vector v is odd then color $(P) = red$

Any other equivalence class E_B of a point B is related by a translation of vector v such that $B = A + v$ and may be colored independently, using the same procedure of parity.

3 Bibliography

1 Soifer A., *The Mathematical Coloring Book*, New York, Springer, 2009.

2 Woodall D. R., *Distances Realized by Sets Covering the Plane*, Journal of Combinatorial Theory (A) **14**, 187-200 (1973)

4 Conclusion

Our proof of $\chi(\mathbb{Q}^2) = 2$ is different from that of Woodall. Our approach is based on rational vectors, while Woodall used rational points, though the arithmetic idea of parity is essentially the same. Our approach seems us more intuitive becasue it could be considered as an extension of the procedure used for coloring the Integer Plane with 2 colors. The key idea was to restrict our study to the set $\mathbb{D}$ of rational numbers whose reduced form have odd denominator.