Avalanche Simulator

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1 The problem

Let us consider a quadratic, two-dimensional grid, composed by $n \times n$ little squares that we call sites, such as a chessboard. In each site of the grid there is a given number of snowflakes, on top of one another. If the number of snowflakes on a site is 4 or greater then the site is unstable. At each instant, an unstable site transfers one snowflake to each of its four nearest neighbors. If the site is on the boundary, the snowflakes falling out of the grid are lost. This rule can generate some kind of avalanches.

2 The model

Each site x of the grid is characterized by its *height* $H(x)$, the number of snowflakes on it. We say that x is *stable* if its height is lesser than 4. Otherwise, $H(x) \geq 4$, and the site x is *unstable*. (Figure 1).

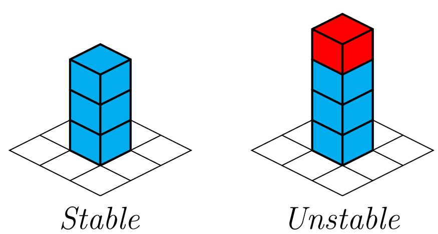


Figure 1: A stable site and an unstable one.

2.1 Von Neumann neighbourhood

The dynamics of the avalanche is local, in the sense that the new value $H(x)$ of a site x depends only on the value of the height H in the sites near x.

So we consider the *von Neumann neighbourhood* of a site x, given by x itself and its four adiacent sites (Figure 2).

Let the 4 neighbouring cells be called n, s, e, w (north, south, east and west). The neighbourhood is thus:

$$N(x) = (x, n, s, e, w)$$

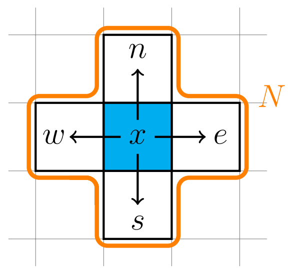


Figure 2: von Neumann neighbourhood.

2.2 States of the system

We are interesting to study the evolution of the system. At each instant the state of the system is given by $\{(x, H(x)) : x \in Grid\}$. A state is *stable* if every site is stable, otherwise it is called *unstable*.

Let us assume that for every site x the initial height $H(x)$ be a number in the set $Q = \{0, 1, 2, \ldots, 7\}$. We can see that, under this hypothesis, the height of every site cannot exceed the value 7 during all the avalanche.

In Figure 3 we can see an example of an initial *state* in a grid 5×5.

1	2	1	3	2
2	4	3	1	3
1	2	2	3	1
4	1	3	2	3
2	3	1	5	2

Figure 3: An example of initial state.

2.3 Dynamics: the local rule

If we know the state of the system at an instant t we can determine the state of a cell at the next instant $t+1$. What we need to calculate the height $H(x)$ on a site x, at instant $t+1$, is the height of site x and its neighbors n, s, e, w at instant t. The dynamic effect is given by a function λ (Figure 4).[1]

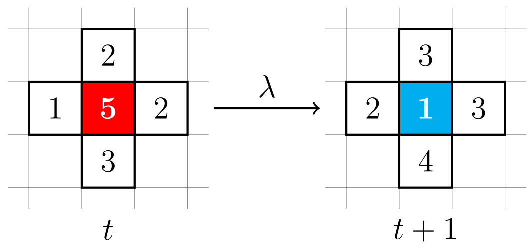


Figure 4: Local rule.

We now see the arithmetics of λ.

The function u tells us whether a cell is stable or not:

$$u(x) = \begin{cases} 0 \text{ if } x \textit{ stable} \\ 1 \text{ if } x \textit{ unstable} \end{cases} \tag{1}$$

We can now write the *local rule* as

$$\lambda(N(x)) = H(x) - 4u(x) + u(n) + u(s) + u(e) + u(w) \tag{2}$$

where we take the height $H(x)$, then we drop out 4 snowflakes if the site is *unstable* (one for each neighbor site), and finally we add as many snowflakes as the number of *unstable* neighbors.

Equation (2) defines the dynamics of our model. We have implemented the geometry of the grid and its dynamics using software *Wolfram Mathematica*.

2.4 Parameters and quantities.

We define parameters and quantities of the model that we'll use in our investigtion.

[1] Note: this diagram is partial, we are neglecting the effect of dynamics on the neighbors.

2.4.1 Size.

A parameter of our model will be given by the side L (*size*) of the square grid. The space is discrete in our model so L is an integer. The *surface* interested by the avalanche is L^2 (Figure 5).

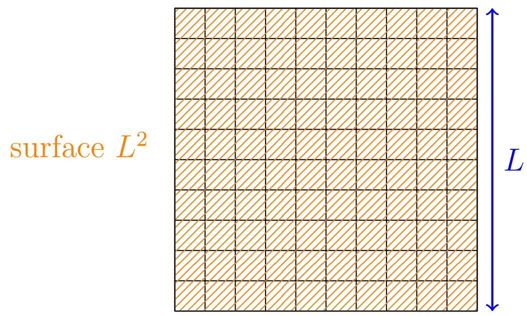


Figure 5: Size and surface.

2.4.2 Initial mass and initial density.

The *initial mass* N_0 is the **total** number of snowflakes on the grid at the instant $t = 0$.
The *initial density* σ_0 is defined as

$$\sigma_0 = \frac{N_0}{L^2}$$

2.4.3 Duration.

Time is discrete and its unit is the step from one state to the next one. The *duration* T of an avalanche is the number of steps from the beginning until the system reaches a stable state.

2.4.4 Final mass and final density.

Final mass N_0 is the **total** number of snowflakes on the grid at the end of the avalanche. *Final density* is defined as $\sigma_f = \frac{N_f}{L^2}$.

2.4.5 Magnitude.

The *magnitude* M of the avalanche counts the number of instability events during the process.

2.4.6 Radius.

The *radius* R of the avalanche is the maximum distance of an unstable site of the avalanche from the center of the grid. We assume the *taxicab metric* to measure the distance between two sites of the grid (Figure 6):

$$\overline{AB} = |x_A - x_B| + |y_A - y_B|$$

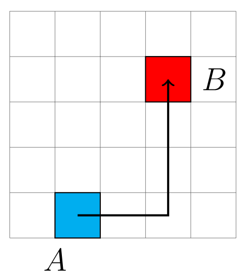


Figure 6: The taxicab metric: $\overline{AB} = 5$.

The Von Neumann neighbourhood of a site x is given by the sites at distance ≤ 1 from x. It is the unitary circle in the taxicab metric.

2.5 A simulation of the model.

We can see an example of avalanche:

1	2	1	3	2
2	4	3	1	2
1	2	2	3	1
4	1	3	2	3
2	3	1	5	2

$t = 0$

$\longrightarrow$

1	3	1	3	2
3	0	4	1	2
2	3	2	3	1
0	2	3	3	3
3	3	2	1	3

$t = 1$

$\longrightarrow$

1	3	2	3	2
3	1	0	2	2
2	3	3	3	1
0	2	3	3	3
3	3	2	1	3

$t = 2$

Values of parameters:

- *size* $L = 5$
- *surface* $L^2 = 25$
- *initial mass* $N_0 = 56$
- *initial density* $\sigma_0 = \frac{N_0}{L^2} = \frac{56}{25} = 2.24$
- At the initial state $t = 0$ there are 3 unstable sites.
- At $t = 1$ there's only one unstable site.
- At $t = 2$ the state is stable and the avalanche ended.

We can measure the quantities that characterize the avalanche:

- *Duration* is 3 steps.
- *Final mass* is $N_f = 54$.
- *Final density* is $\frac{N_f}{L^2} = \frac{54}{25} = 2.16$,
- *Magnitude* is 4, the number of sites became unstable during the evolution: 3 at $t = 0$ and 1 at $t = 1$.
- *Radius* is 7, the maximum distance of unstable sites from the center of the grid.

2.6 Initial conditions

We have simulate the initial distribution of the snowflakes after a random snowfall. We have done the following assumptions:

- side L of the square grid is fixed to 16
- the number N_0 of snowflakes fallen on the grid is randomly choice such that initial density σ_0 is between 1 and 3.5.
- each snowflake can fall on each site of the grid with uniform probability

We found that under these hypothesis the *number of snowflakes on a site* has the distribution of probability:

$$0 \to 0.17, 1 \to 0.27, 2 \to 0.25, 3 \to 0.17, 4 \to 0.09, 5 \to 0.04, 6 \to 0.01, 7 \to 0$$

3 Results

We have done a series of experiments on a computer using the software Wolfram Mathematica 9. In each experiment we measured two quantities or parameters X, Y of the avalanche. Let us assume that data $(X_1, Y_1), \ldots, (X_n, Y_n)$ are available, and that we suspect it exists a relationship between X and Y, so that $Y = f(X)$. We must be conscious that $Y = f(X)$ could not be exactly satisfied by individual data but is a statistical law. It involves the mean values $\bar{X}, \bar{Y}$ of the quantities X, Y.

3.1 Initial mass and final mass

We have studied relation between *initial mass* N_0 and *final mass* N_f in an avalanche.

- Assuming that initial density has a fixed value $\sigma_0 = 2.4$ and size L of the grid is variable we found a relation as

  $$N_f = N_0(AL^k + B)$$

  with $A = 0.26, B = 0.06, k = -0.8$.
  We could call B the *factor of dissipation*. In fact, as size L increases B approximates the ratio $\frac{N_f}{N_0}$.

- Next we have assumed that size of the grid has a fixed value $L = 12$ and *initial density* σ_0 is variable. We found that ratio $\frac{N_f}{N_0}$ increases from 0 to 0.11 when σ_0 runs from 0 to 2.7, while it remains stable in the interval [0.11, 0.12] for $2.7 \leq \sigma_0 \leq 5$.

- We have repeated the experiment assuming other fixed values for the size of the grid, $L = 11, 13, 14, 15$. We found that the relation between σ_0 and the dissipative factor $\frac{N_f}{N_0}$ has the same behavior: it increases until it reaches value 0.11 when $\sigma_0 = 2.7$ and remains in the interval [0.11, 0.12] for $2.7 \leq \sigma_0 \leq 5$.

3.2 Initial density σ_0 and duration T

We have studied the relationship between *initial density* σ_0 and *duration* T in an avalanche. Assuming that size L of the grid is fixed, we have searched for a relation of the form

$$T = A\sigma_0^k + B$$

We have experimented for ten values of L between 5 and 30 and we have found a surprising correlation:

$$T = A\sigma_0 + B$$

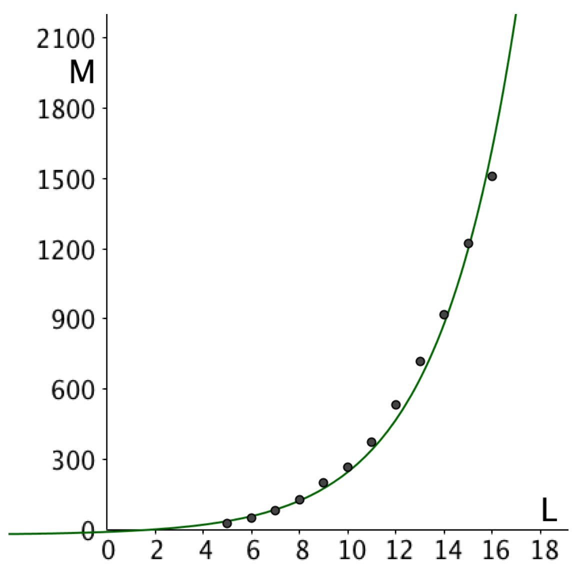
the relation between T and σ_0 is independent of size L of the grid. The relation is linear with costants $A = 5.0$, $B = -5.3$.

3.3 Size, magnitude and initial density

Let us focus on the relationship between *initial density* σ_0, size L and *magnitude* M that counts the number of unstable sites in the avalanche.

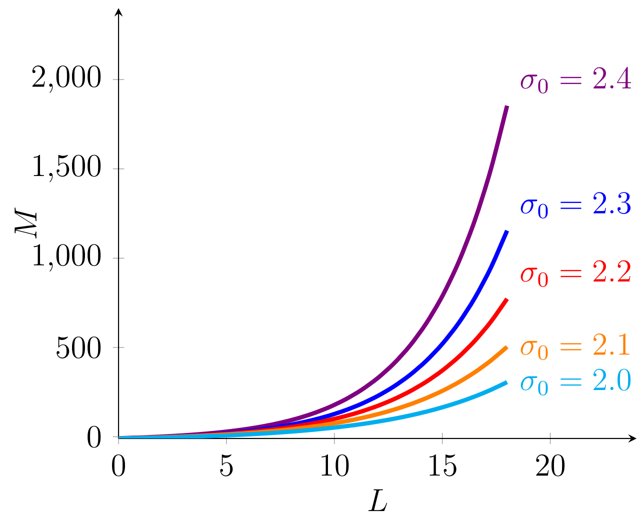
- In the first experiment we fixed the value $\sigma_0 = 2.6$ of the initial density and we searched for a relation between size L, running from 5 to 20, and magnitude M. Data said that M increases very fast as L increases, so we searched for an exponential law:

$$M = Ae^{kL} + B$$



Making use of *software Geogebra* we found that $A = 11.72, B = -21.10, k = 0.315$ gives an exponential law that well fits the data.

- In the next step, we let initial density σ_0 run in the range of values $\{2.0, 2.1, \ldots, 3.0\}$. For each value of σ_0 we found an exponential function with parameters A, B, k.



While σ_0 increases of 0.1 we observed that

  - parameters A, B remains approximately constant,
  - k increases of an approximately constant amount

- Data suggests the existence of a constant α such that linear relation $k = \alpha\sigma_0$ holds.

- so, we found an exponential law

$$M = Ae^{\alpha\sigma_0 L} + B$$

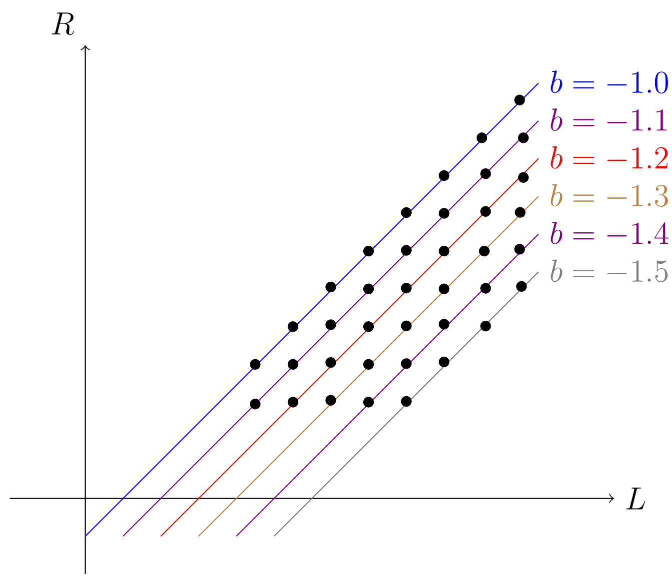
that that connect all together L, σ_0, M.

3.4 Size, radius and initial density

We have studied the relationship between *initial density* σ_0, size L and *radius* R that measure the maximum distance of an unstable site in the avalanche.

- In the first experiment we fixed the value $\sigma_0 = 1.8$ of the initial density and we searched for a linear relation between size L, running in the set $\{7, 9, 11, 13, 15, 17, 19\}$, and radius R.

$$R = aL + b$$



Making use of *software Geogebra* we found that $a = 1, b = -1.5$ gives a linear model that well fits the data.

- In the next step, we let initial density σ_0 run in the range of values $\{1.9, 2.02.1, \ldots, 2, 9\}$. For each value of σ_0 we found that the linear function $R = aL + b$ has always coefficient $a = 1$ while coefficient b increases from -1.5 to -1.

- For $\sigma_0 \geq 2.7$ we found linear function $R = L - 1$. As expected, size L is limited by

$$R \leq L - 1$$

and while initial density σ_0 growths we have that R growths until it reaches its limit $L - 1$. This result just reflects the geometry of the grid and the intuitive fact that in an avalanche snowflakes will reach the boundary of the grid if their number is great enough.

4 Conclusions

Our model of avalanche belongs to the family of *cellular automata*[2], introduced in the 1940s by Stanislaw Ulam and John von Neumann. We have done some choices for the set of parameters and quantities that we think are characteristic of the avalanche. After we have collected data through a series of case studies by computer.
Modeling aspects discussed in our research can be understood with very little knowledge on statistical methods and mathematics. Each experiment required the random choice of thousand data, limited by finite computer resources. The best approximation to a given data set was determined using the tools of software *Mathematica* and *Geogebra*. A visual analysis of the data involves some freedom, so that different observers will not agree on the results obtained from the same data set. Sometimes the analysis simply reduces to transferring the data to the computer and move some sliders, i. e. the parameters of the model, with the mouse. We are well conscious that our results may be affected by improper use of statistical treatment of the data. Our aim was simply to demonstrate that a mathematical model exists and could be found even lacking in precision. No matter which model we are using, we always look real world through a finite window, and finite data set must contain both a lower and an upper limit of object sizes. The line followed here is somewhat naive, it just describes how every student would work starting from some basic knowledge on mathematics with some good ideas.

[2] `http://mathworld.wolfram.com/CellularAutomaton.html`