Tropical Mathematics

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Overview

Problem

Let's define two new operations: $a \oplus b := min\{a; b\}$ and $a \odot b := a + b$, for example: $3 \oplus 7 = 3$, $3 \odot 7 = 10$. What are the properties of these new operations? Can we "do anything", as with the addition and multiplication we are used to? Can we define the subtraction and the division?

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Overview

1. What Arithmetics? (number sets, operations and their properties)
2. What Algebra? (polynomials and functions, factorizations, roots)
3. What Geometry? (planar algebraic curves)

First questions

- ▷ What numbers can we eventually build within this arithmetics?
  ▷ What numbers do we consider?

- ▷ Can we always consider $\cdot$ an iteration of $+$, and $^\wedge$ of $\cdot$?
  $2 \cdot 3 = 2 + 2 + 2$, but $2 \cdot \sqrt{5} =?$
  ▷ What operations can we define?

- $-2,\ \frac{1}{2},\ \sqrt{2}, \ldots$
  ▷ Do these symbols have the same meaning in tropical arithmetics?

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In $\mathbb{N}$, neither $\oplus$ nor $\odot$ admits inverse elements. Can we add them?

- $\oplus$ **NO**: $2 \oplus 3 = 2$, $2 \oplus 4 = 2$, hence $2 \ominus 2 =$?! is it $3, 4, \ldots$?
- $\odot$ **YES**:

  $\odot n := +n := [(n, 0)] := \{(n, 0), (n+1, 1), (n+2, 2), \ldots\}$

  $\odiv n := -n := [(0, n)] := \{(0, n), (1, n+1), (2, n+2), \ldots\}$

  $[(a, b)] \odiv [(c, d)] := [(a, b)] \odot (\odiv [(c, d)]) = [(a \odot d, b \odot c)]$

Comparison between Standard and Tropical Arithmetics

Standard Arithmetics

Tropical Arithmetics

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Standard Arithmetics

$(\mathbb{N}, +, \cdot), \wedge$

Tropical Arithmetics

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Standard Arithmetics | *Tropical Arithmetics*

$(\mathbb{N}, +, \cdot), \wedge$ ⋯→ $(\mathbb{N}, \oplus, \odot), \oslash$

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Standard Arithmetics

$(\mathbb{N}, +, \cdot), \wedge$

inverses $+$

$(\mathbb{Z}, \overset{-}{+}, \cdot), \wedge$

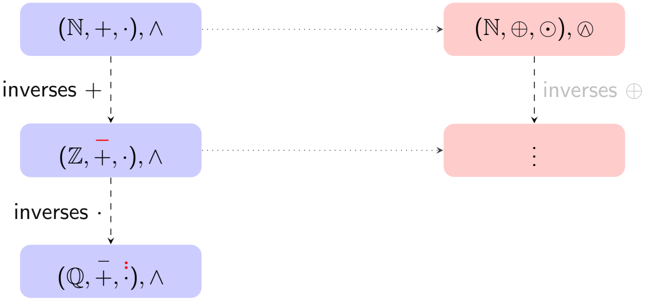
Tropical Arithmetics

$(\mathbb{N}, \oplus, \odot), \owedge$

Comparison between Standard and Tropical Arithmetics

Standard Arithmetics		*Tropical Arithmetics*
$(\mathbb{N}, +, \cdot), \wedge$	→	$(\mathbb{N}, \oplus, \odot), \owedge$
↓ inverses $+$		↓ inverses $\oplus$
$(\mathbb{Z}, \overset{-}{+}, \cdot), \wedge$	→	$\vdots$

Standard Arithmetics *Tropical Arithmetics*



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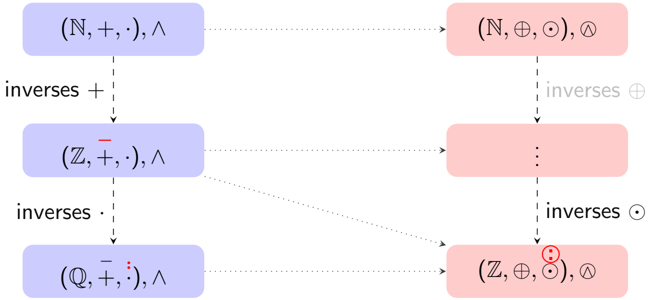

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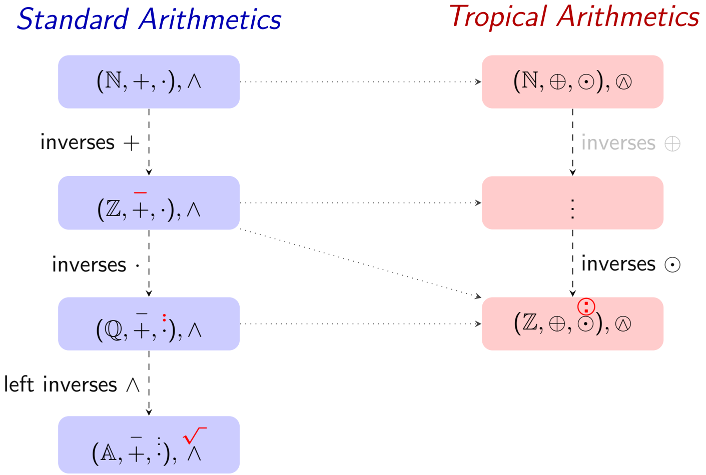

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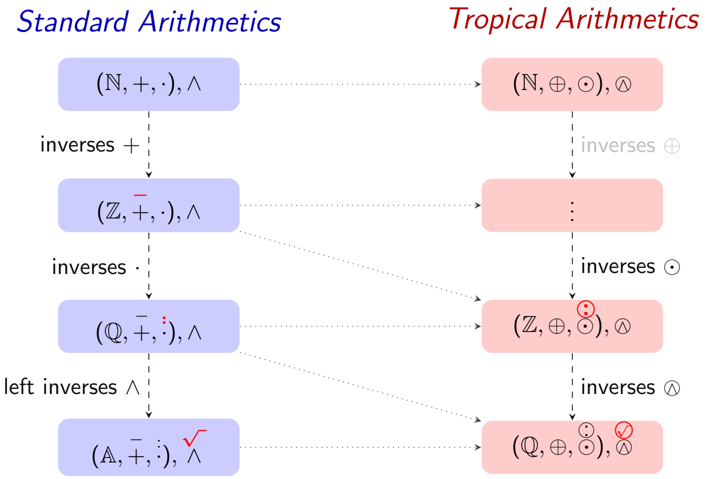

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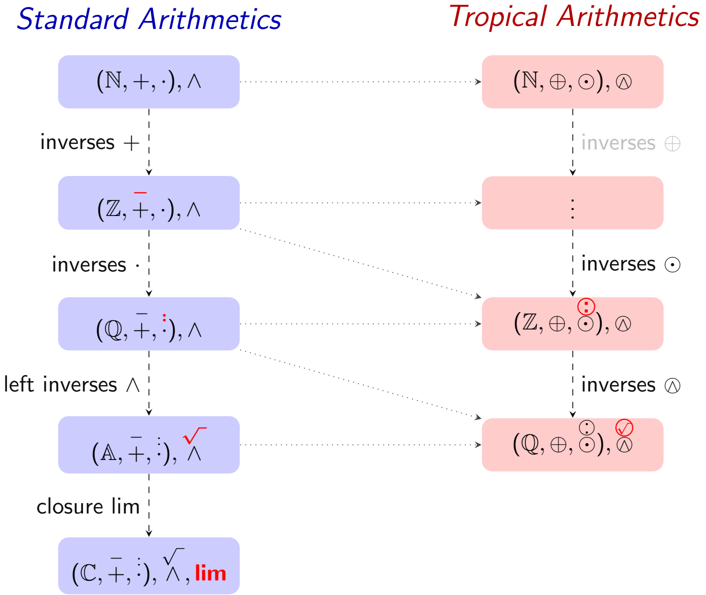

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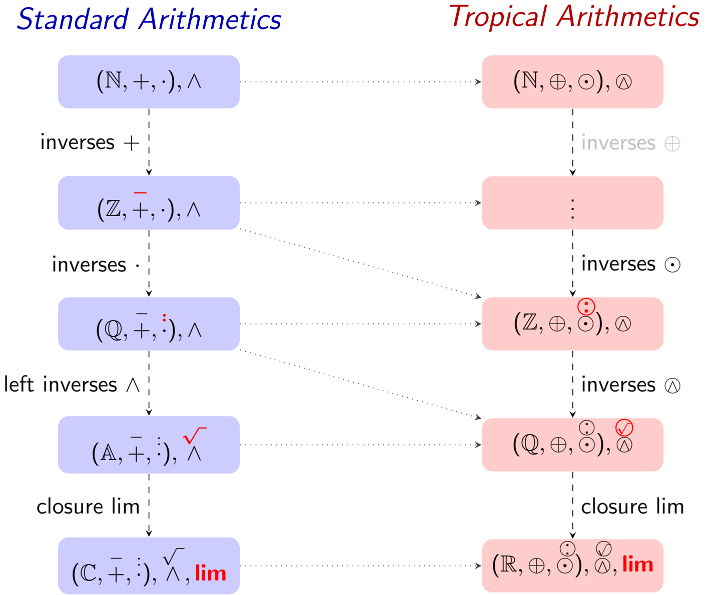

Properties in $(\mathbb{R}, \oplus, \otimes)$

PROPERTY	$\bigoplus$	$\bigodot$
Commutative	✓	✓
Associative	✓	✓
Dissociative	✓	✓
Neutral element	X	✓
Symmetric element	X	✓
Distributive	✓	

Polynomials

By combining numbers and variables with the operations $\oplus$, $\odot$, we get two objects:

$$\textit{sums of products} \quad \overset{\text{distrib}}{\longleftarrow} \quad \textit{products of sums}$$

$$(1 \odot x \odot x) \oplus (2 \odot x) \oplus 0 \quad \xrightarrow{\text{distrib, FTA}} \quad (0 \oplus x \oplus x) \odot (2 \oplus x) \odot 1$$

Definition

A tropical **monomial** is a tropical product of numbers and variables, where repetitions are allowed.
A tropical **polynomial** is a tropical sum of tropical monomials.

Example:

$$2 \odot x^{\odot 4} \odot y^{\odot 2} \oplus 4 \odot x \odot y^{\odot 6}$$

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Tropical polynomial function

To every polynomial p we associate a function
$f_p : \mathbb{R}^n \longrightarrow \mathbb{R} : \; y = p(x_1, \ldots, x_n)$

$$p(x) = \quad -1 \odot x^{\otimes 2} \quad \oplus \quad 0 \odot x \quad \oplus \quad 3$$

$$\min\{ \; -1 + 2 \cdot x \; , \; 0 + 1 \cdot x \; , \; 3 + 0 \cdot x \; \}$$

$$f_p(x) = \begin{cases} 2 \cdot x - 1 & \text{if } x < 1 \\ x & \text{if } 1 \leq x < 3 \\ 3 & \text{if } 3 \leq x \end{cases}$$

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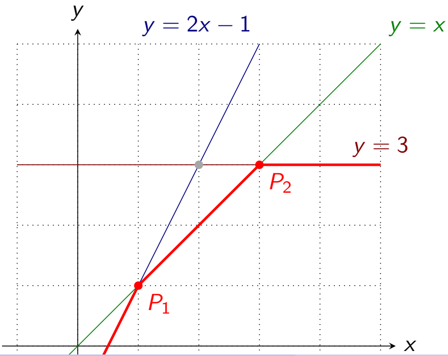

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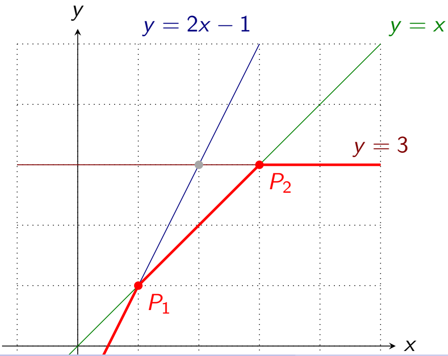

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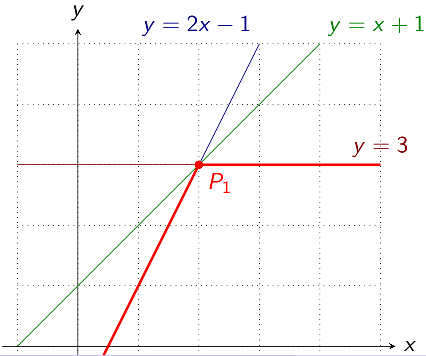

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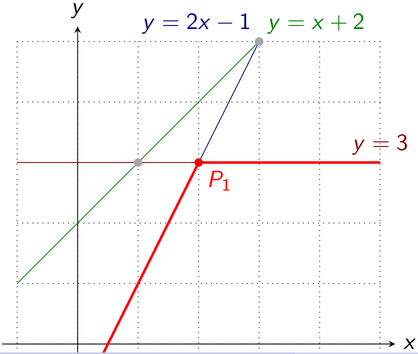

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Equality between polynomials

$\Longrightarrow$ the correspondence $p \longrightarrow f_p$ is **not** injective

Contrarily to the standard algebra ☹

How can we obtain this? By collecting polynomials associated to the same polynomial function in equivalence classes:

Classes of equivalence

$$[a \odot x^{\oslash 2} \oplus c] = \{a \odot x^{\oslash 2} \oplus b \odot x \oplus c \mid b^{\oslash 2} \geq a \odot c\}$$

$$[a \odot x^{\oslash 2} \oplus b \odot x \oplus c] = \{a \odot x^{\oslash 2} \oplus b \odot x \oplus c \mid b^{\oslash 2} < a \odot c\}$$

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Factorization

Sum of products $\dashrightarrow$ product of sums

(suppose $a \neq +\infty$)

1 $\quad a \odot x \oplus b = a \odot (x \oplus b \odiv a)$

2 $\quad a \odot x^{\owedge 2} \oplus b \odot x \oplus c =$

$= a \odot (x \oplus b \odiv a) \odot (x \oplus c \odiv b) \quad \text{if} \quad b^{\owedge 2} < a \odot c$

$= a \odot (x \oplus (c \odiv a) \oslash 2) \quad \text{if} \quad b^{\owedge 2} \geq a \odot c$

$\vdots$

$n \quad a_n \odot x^{\owedge n} \oplus \ldots a_1 \odot x \oplus a_0$

$= a_n \odot (x \oplus (a_{n-1} \odiv a_n)) \odot \ldots (x \oplus (a_{n-k-1} \odiv a_{n-k})) \odot \ldots \odot (x \oplus (a_0 \odiv a_1))$

Fundamental Theorem of Algebra

The tropical semiring $(\mathbb{R}, \oplus, \odot)$ is algebraically closed.

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Roots

- In standard arithmetics:

$p(x) = a \cdot x^2 + b \cdot x + c = a \cdot (x - x_1) \cdot (x - x_2)$
where x_1 and x_2 have both the property $p(x_i) = 0$ and are called **roots** (zeros) of p

- What happens in the tropical arithmetics?

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x_1 and x_2 are the values in which at least two monomials take the same value and, at the same time, are the least of the polynomial.

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Given a tropical polynomial $p(x)$, a **tropical root** is a number c for which $p(c) = a_i \odot c \owedge n_i = a_j \odot c \owedge n_j$ for at least two indexes $i \neq j$.

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where x_1 and x_2 have both the property $p(x_i) = 0$ and are called **roots** (zeros) of p

- What happens in the tropical arithmetics?

Again $p(x) = a \odot x^{\owedge 2} \oplus b \odot x \oplus c = a \odot (x \oplus x_1) \odot (x \oplus x_2)$
but now, neither $p(x_i) = 0$ nor $p(x_i) = +\infty$:

$$p(x) = -1 \odot x^{\owedge 2} \oplus x \oplus 3 = -1 \odot (x \oplus \mathbf{1}) \odot (x \oplus \mathbf{3})$$

$x_1 = 0 \odot -1 = \mathbf{1}$ $\quad p(1) = -1 \odot 1^{\owedge 2} \oplus 1 \oplus 3 = \min\{-1 + 1 \cdot 2, 1, 3\} = \min\{\boxed{1}, \boxed{1}, 3\} = \boxed{1}$

$x_2 = 3 \odot 0 = \mathbf{3}$ $\quad p(3) = -1 \odot 3^{\owedge 2} \oplus 3 \oplus 3 = \min\{-1 + 3 \cdot 2, 3, 3\} = \min\{5, \boxed{3}, \boxed{3}\} = \boxed{3}$

x_1 and x_2 are the values in which at least two monomials take the same value and, at the same time, are the least of the polynomial.

Tropical Roots

Given a tropical polynomial $p(x)$, a **tropical root** is a number c for which $p(c) = a_i \odot c \owedge n_i = a_j \odot c \owedge n_j$ for at least two indexes $i \neq j$.

Geometric loci

In the traditional maths a *geometric locus* is the subset of points in space that satisfy a particular property (e.g. points of coordinates (x, y) satisfying some equations $y = f(x)$ or being the root of some function $F(x, y) = 0$).

Tropical Algebraic Curve

A **tropical algebraic curve** in $\mathbb{R}^n$ is the geometric locus of the tropical roots of a tropical polynomial $p(x_1, \ldots, x_n)$.

Example:
A tropical polynomial in **two** variables x, y:

$$p(x, y) = -1 \odot x^{\oslash 2} \oplus x \oplus 3$$

gives rise to a tropical curve in the Real plane $\mathbb{R}^2$,
that is two lines $x = 1$ and $x = 3$ (roots).

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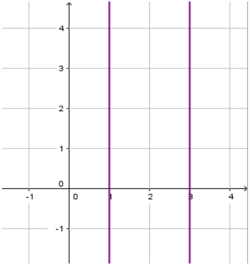

Examples: from tropical polynomials to algebraic curves

$$p(x,y) = x^{\oslash 2} \oplus y^{\oslash 2} \oplus 1$$

$$\min\{\underbrace{x \cdot 2}_{A},\ \underbrace{y \cdot 2}_{B},\ \underbrace{1}_{C}\}$$

$A = B \leq C$

$$\begin{cases} 2 \cdot x = 2 \cdot y \\ 2 \cdot x \leq 1 \end{cases} \Longrightarrow \begin{cases} y = x \\ x \leq \frac{1}{2} \end{cases}$$

$C = B \leq A$

$$\begin{cases} 1 = 2 \cdot y \\ 1 \leq 2 \cdot x \end{cases} \Longrightarrow \begin{cases} y = \frac{1}{2} \\ x \geq \frac{1}{2} \end{cases}$$

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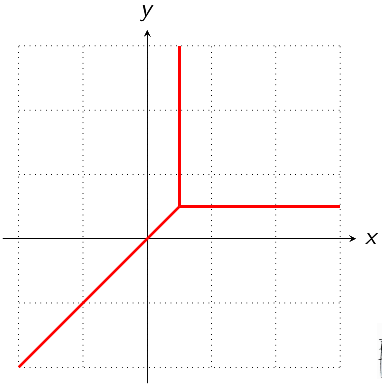
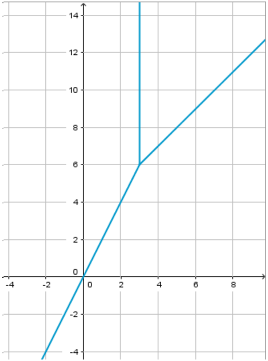


Examples: from tropical polynomials to algebraic curves

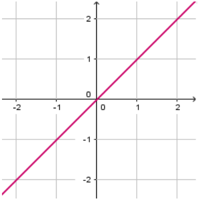
$p(x, y) = x^2 \oplus 3 \otimes x \oplus y$

"tropical parabola"



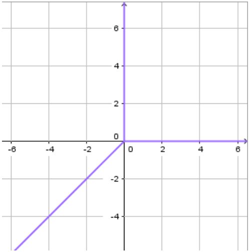
$p(x, y) = x \oplus y$

"tropical line"



$p(x, y) = x \oplus y \oplus 0$

"tropical line"



Thank you for your attention!