Le carreleur

2015-2016

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1 The problem

A tiler has to tile all the rooms in a house. For the first room, he has identical square tiles. For the second room the tiles are hexagons. How does he arrange them to cover the whole, infinitely wide floor of each room ? He shouts for desperation when he discovers that for the third room he has to use pentagonal tiles. Why ? Can you help him to arrange the tiles in order to cover the widest surface possible ?

2 Conjectures and results

We found a tiling of regular pentagons (Figure 1) conjectured as an optimal solution of the problem (Y. Limon Duparcmeur, A. Gervois, J. Troadec. Dense Periodic Packings of Regular Polygons. Journal de Physique I, EDP Sciences, 1995, 5 (12), pp.1539-1550.). We have proved that this tiling has the lowest density among all the tilings that possess a special symmetry.

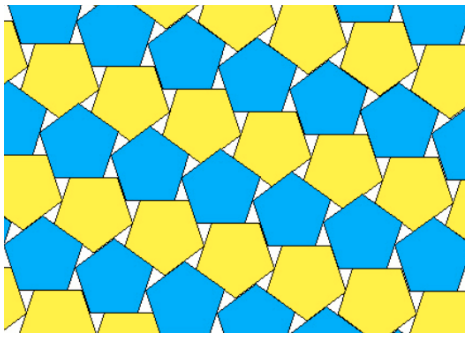

FIGURE 1 –

The symmetry of the conjectured optimal tiling (Figure 1) has drived us to concentrate our study upon a set of tilings that satisfy a special symmetry. Another argument in favour of symmetry is that it helped us in the main difficulty of this optimization problem that is the number of parameters we have to treat. It seems impracticable also with computers, and symmetry has revealed its power in reducing the number of parameters.

3 Tiling the plane

A *tessellation* or *tiling* is a covering of the plane by infinity copies of the same tile, without gaps or overlappings.

3.1 Tessellations and symmetries

Problem 1. Can we tessellate the plane with a given triangle T as tile?
Yes, and a solution uses a symmetry centered in the midpoint of a side of triangle T (Figure 2)

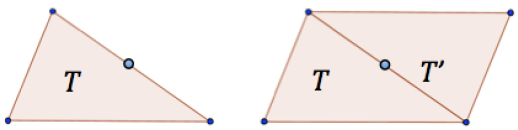

FIGURE 2 –

We obtain a new polygon $P = T \cup T'$, a parallelogram, and we know that we can tile the plane with P, the *pattern* of the tiling (Figure 3).

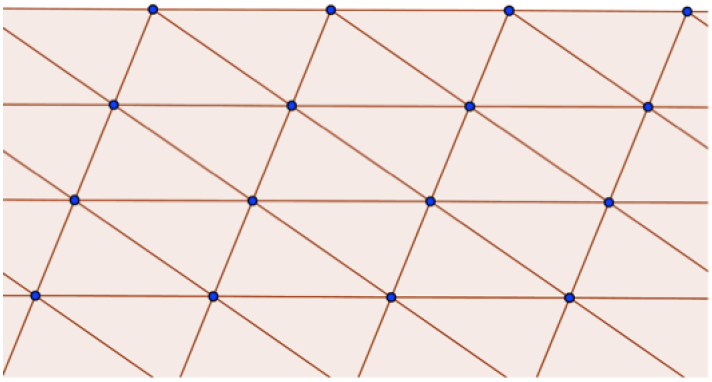

FIGURE 3 –

Problem 2. Can we tessellate the plane with a quadrilateral Q as tile?
We repeat the idea of symmetry with center in the midpoint of a side of Q (Figure 4).

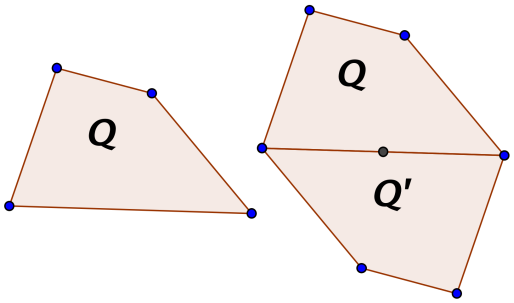

FIGURE 4 –

We obtain a a special hexagon $E = Q \cup Q'$ that can be used to tile the plane (Figure 5).

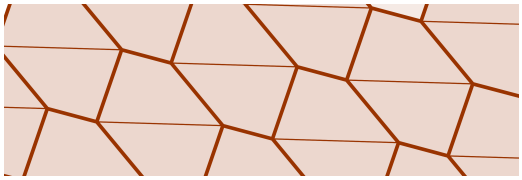

FIGURE 5 –

3.2 A symmetric pattern of pentagons.

Note that the translations of the tessellation generateted by a symmetric pattern are determined by the vertices of pattern (Figure 6).

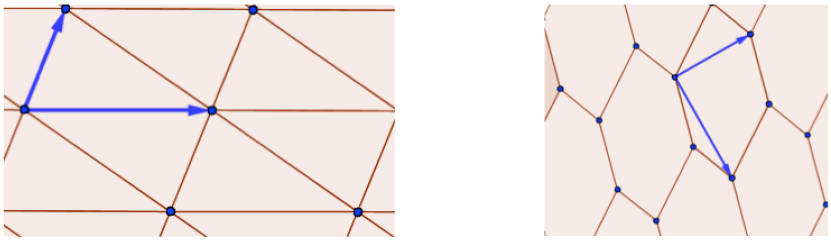

FIGURE 6 –

Given a regular pentagon P and a point W let us denote as $P[W]$ the symmetric of P by center W. There are three possible cases.
1) First case : W is exterior to pentagon P (Figure 7). The pentagon tiling is not efficient for the presence of large gaps, so we can exclude this case from our search.

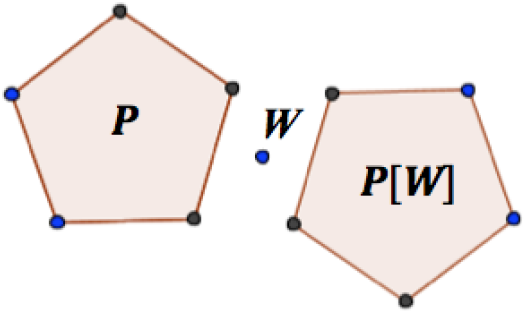

FIGURE 7 –

2) Second case : W is interior to pentagon P (Figure 8). The tiling is not admissible.

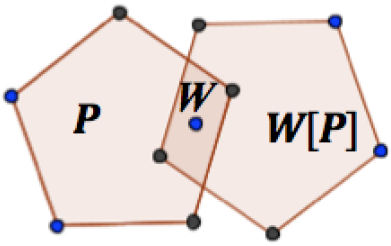

FIGURE 8 –

3) We can restrict us to "Third case" : W is on the perimeter of pentagon P (Figure 9).

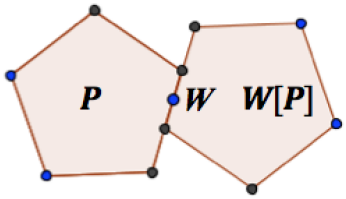

FIGURE 9 –

3.3 Patterns and translations

Given a regular pentagon P let us assume that point W is on a side of P. We'll study tessellations generated by pattern $P[W]$ and two translations t, u (Figure 10).

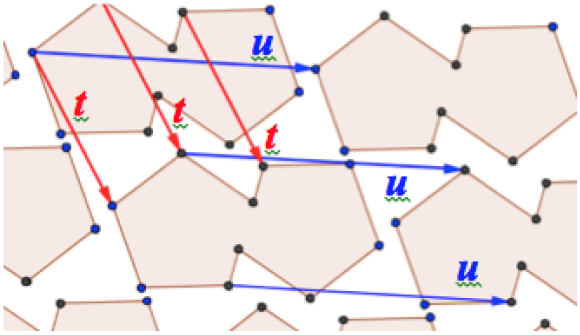

FIGURE 10 –

First, we consider the 1-dimensional tessellation generated by a translation t and a pattern $P[W]$. Let us distinguish three cases.

1) First case (Figure 11). The tiling is not efficient, so we'll not examine this case.

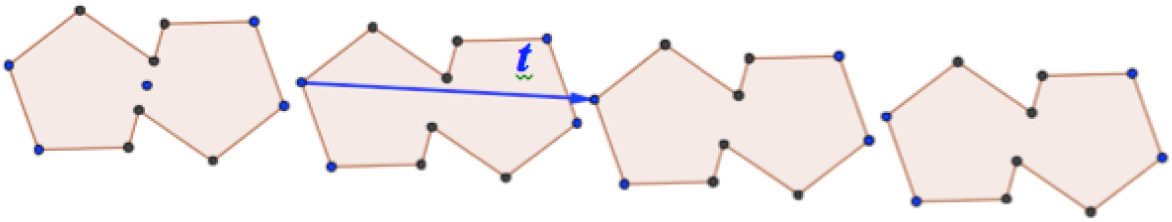

FIGURE 11 –

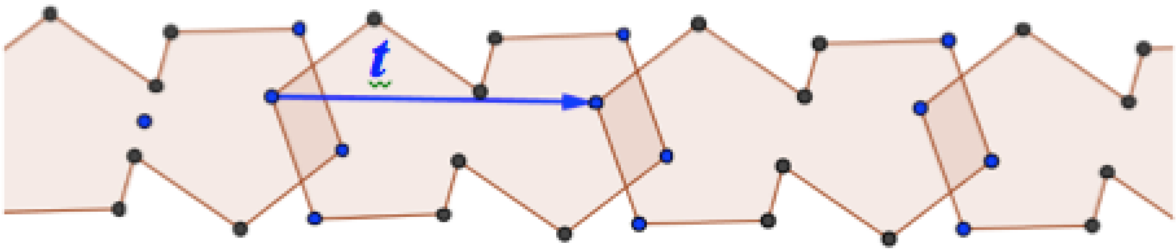

FIGURE 12 –

2) Second case (Figure 12). We discharged also this case, becuase it s not admissible.
3) Third case (Figure 13). We'll restrict our attention to this case.

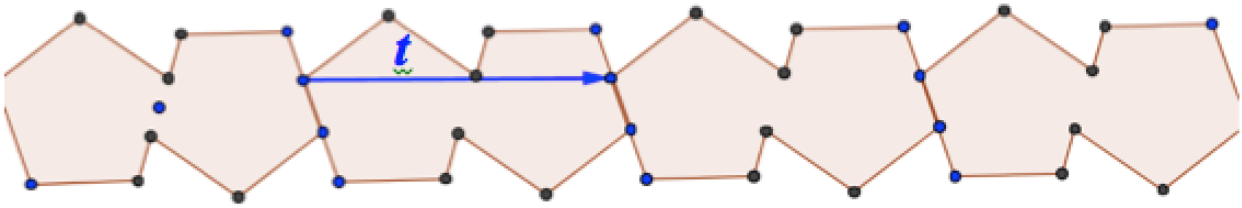

FIGURE 13 –

Now, we classifiy three *forms of contact* between pattern $P[W]$ and its translate $P[W]+t$

Definition 1. We say that translation t determines a *simple touch* on vertex A of pattern $P[W]$ if A lies on a side of $P[W]+t$ (Figure 14).
As a consequence of symmetry a vertex B' of $P[W]+t$ will be on a side of $P[W]$.

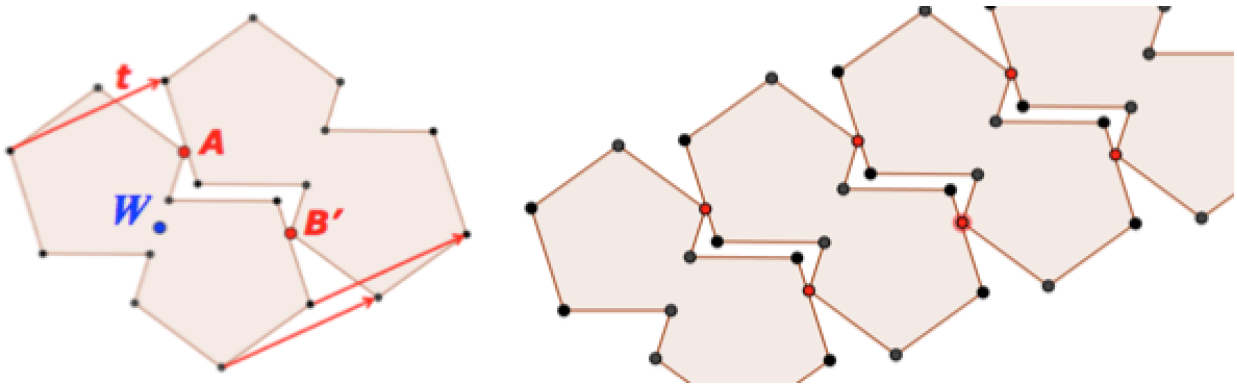

FIGURE 14 –

Definition 2. We say that translation t determines a *weak contact* if one side of $P[W]$ and a side of $P[W]+t$ partially overlap (Figure 15).

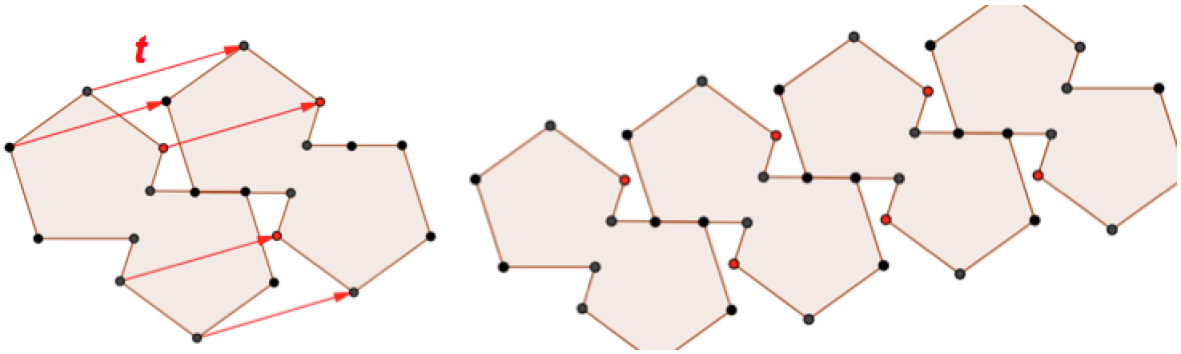

FIGURE 15 –

Definition 3. We say that translation t determines a *strong contact* if one side of $P[W]$ and a side of $P[W]+t$ partially overlap an there is a simple touch in a point out of the overlapping side (Figure 16).

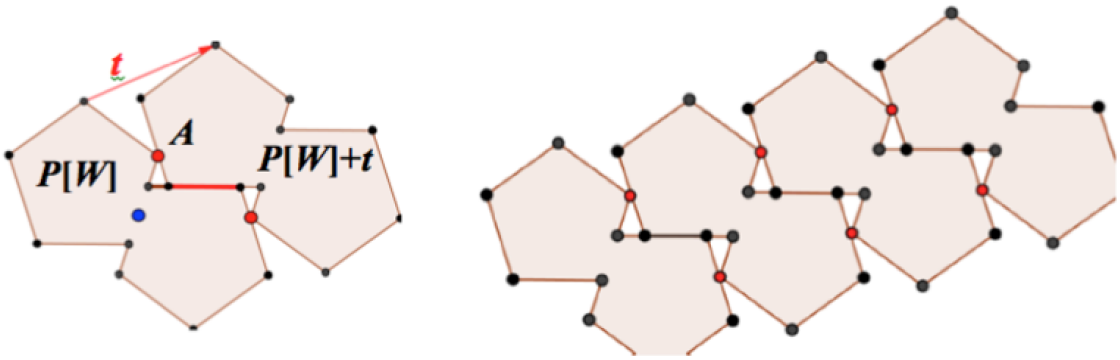

FIGURE 16 –

A fundamental property for our next investigations is

Theorem 1. Given a regular pentagon P and a point W on one side of P, there is a unique translation t such that $P[W], P[W]+t$ have a strong contact.
We have given a proof of the theorem using software *Wolfram Mathematica*.

Problem. Given a pattern $P[W]$ let t be the unique translation such that $P[W]$ and $P[W]+t$ have a strong contact. Do a translation u exist such that $P[W]+u$ has a strong contact with both $P[W]$ and $P[W]+t$?

Our answer to the problem was found using software *Wolfram Mathematica*.

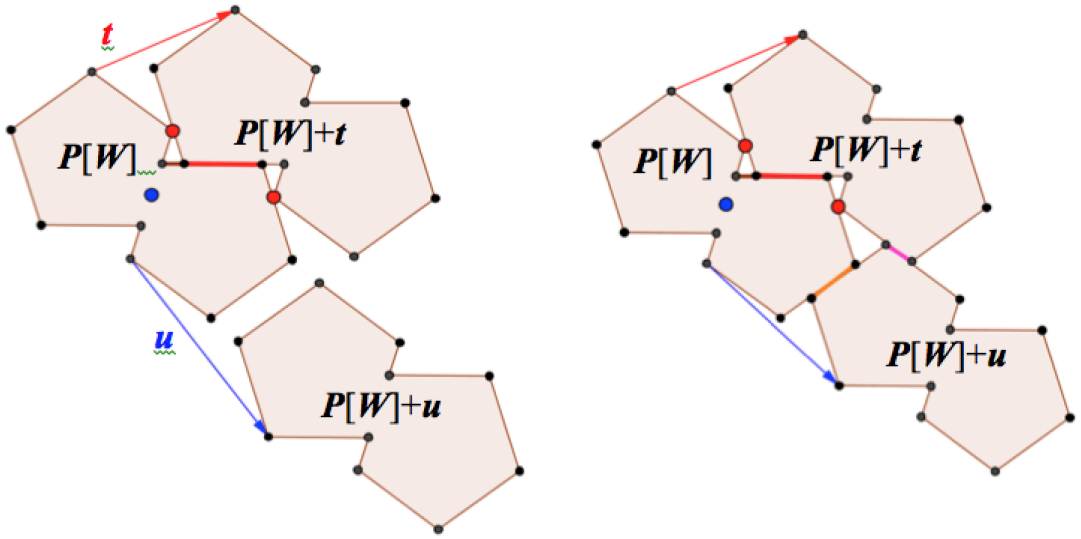

FIGURE 17 –

Theorem 2 Given a regular pentagon P and a point W, for every translation t such that patterns $P[W], P[W]+t$ are in strong contact there exists a unique translation u such that pattern $P[W]+u$ is in strong contact with both $P[W]$ and $P[W]+t$.

3.4 The search for the optimal pattern

Problem. Given a regular pentagon P, find a point W on one side of P such that pattern $P[W]$ generates the tessellation with maximum density (Figure 18).

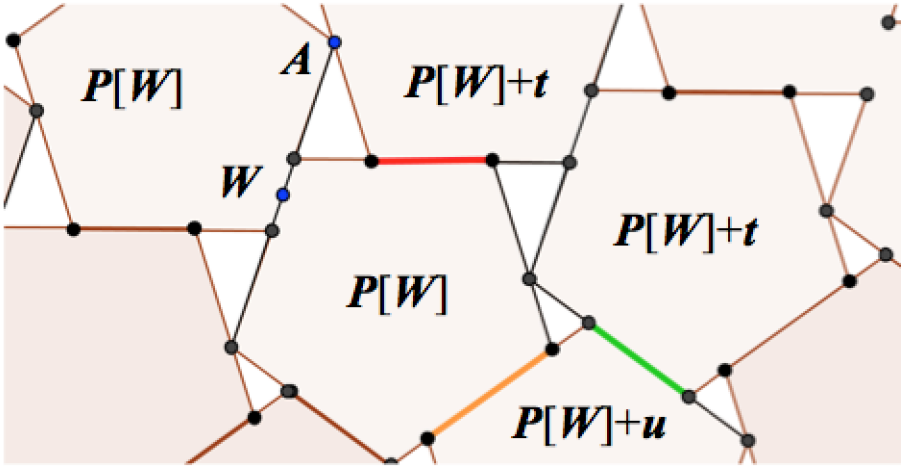

FIGURE 18 –

We can put the problem in a more symmetric form. Let us consider the symmetric dodecagon $S[W]$ (green in Figure 19)

$$S[W] = P[W] \cup H[W]$$

where $H[W]$ is the part of $S[W]$ complementary to $P[W]$ formed by two symmetric pairs of triangles (Figure 19). The problem is to find point W such that $H[W]$ has minimum area.

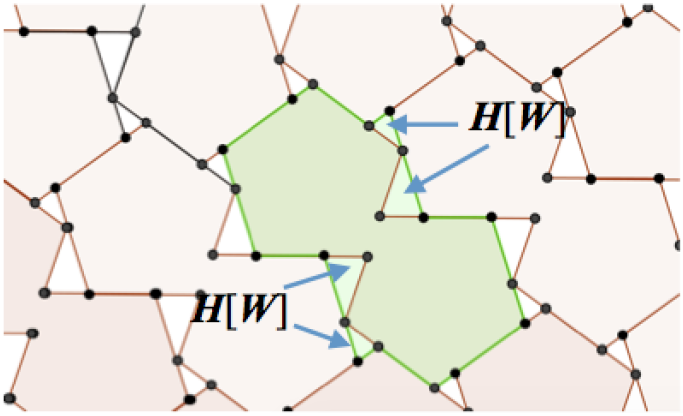

FIGURE 19 –

$H[W]$ is a symmetric polygon, so we can restrict to his half formed by the two triangles ABC, ADE (Figure 20).

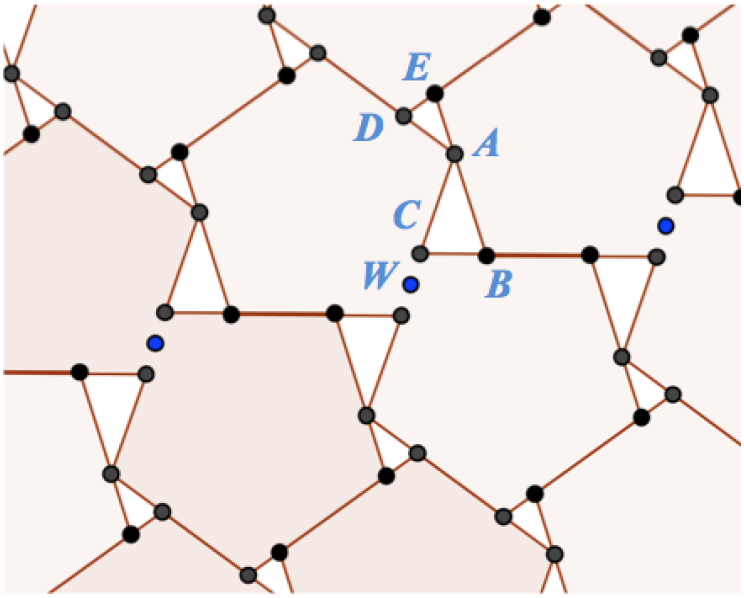

FIGURE 20 –

ABC is an isosceles triangle. In fact, the two angles at vertices B and C are supplementary of the angle of the pentagon (Figure 21). The same argument shows that ADE is a similar isosceles triangle.
Let L be the side of the regular pentagon. Position of point A is determined by a number k, $0 \leq k \leq 1$, such that

$$AB = kBE = kL$$

and

$$AE = (1-k)BE = (1-k)L$$

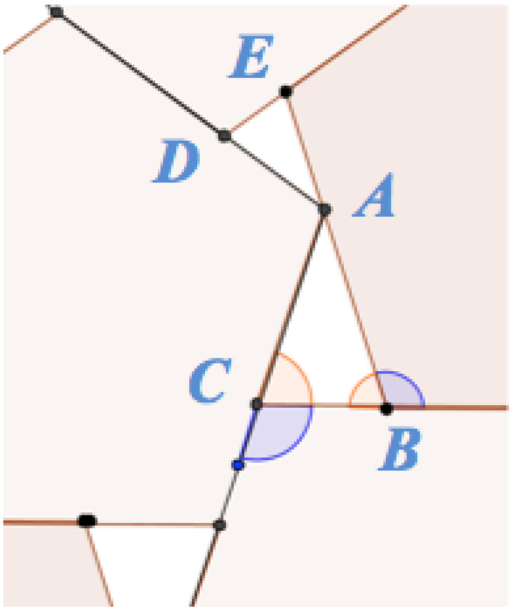

FIGURE 21 –

There is a constant ω such that the area of isosceles triangle ABC is $\omega k^2 L^2$ and the area of isosceles triangle ADE is $\omega(1-k)^2 L^2$. So polygon $H[W]$ has area $2(\omega k^2 L^2 + \omega(1-k)^2 L^2) = 2\omega L^2(k^2 + (1-k)^2)$ This area takes its minimum value, which is ωL^2, when $k = \frac{1}{2}$ i. e. when A is the midpoint of the pentagon side.

4 Conclusion

Translations t, u are uniquely determined by point W (theorems 1, 2) Note that in the optimal solution t takes A to the opposite vertex of pentagon with a direction that is perpendicular to the contact side, while u is perpendicular to t (Figure 22, 23).

The density of the optimal tiling is given by the ratio $\frac{area(H[W])}{area(S[W])} = \frac{5-\sqrt{5}}{3}$

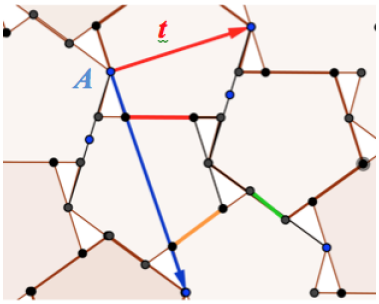

FIGURE 22 –

The tessellation we have found is optimal in the restricted set of "symmetric" tessellations we have examined. Symmetry was required in order to guarantee the periodicity of tessellation. The contact conditions “side to side” or “vertex to side” were translated in linear conditions in our *Mathematica* proofs of theorems 1 and 2.

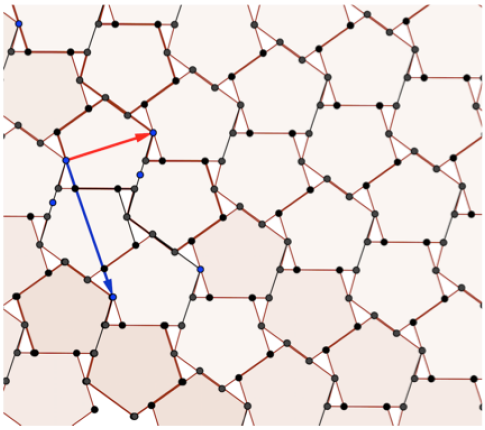

FIGURE 23 –